

A GENERAL FRAMEWORK FOR VARIATIONAL CALCULUS ON WIENER SPACE

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Abstract: We provide a framework to derive a variational formulation for $-\log \mathbb{E}_\nu [e^{-f}]$ for a large class of measures ν . We use a family of perturbations of the identity (W^u) whose invertibility we characterize thanks to entropy. This yields results of strong existence for various stochastic differential equations. We also discuss the attainability of the infimum in the variational formulation and we derive a Prékopa-Leindler theorem for the measure ν .

Keywords: Wiener space, variational formulation, entropy, invertibility, Brownian bridge, loop measure, diffusing particles, stochastic differential equations, Prékopa-Leindler theorem

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1. Introduction

Denote \mathbb{W} the space of continuous functions from $[0, 1]$ to \mathbb{R}^n and \mathbb{H} the associated canonical Cameron-Martin space of elements of \mathbb{W} which admit a density in L^2 . Also denote μ the Wiener

measure, W the coordinate process, and (\mathcal{F}_t) the canonical filtration of W completed with respect to μ . W is a Brownian motion under μ . Set f a bounded from above measurable function from \mathbb{W} to \mathbb{R} . In [5], Dupuis and Ellis prove that

$$(1.1) \quad -\log \mathbb{E}_\mu [e^{-f}] = \inf_{\theta} (\mathbb{E}_\theta [f] + H(\theta|\mu))$$

where the infimum is taken over the probability measures θ on \mathbb{W} which are absolutely continuous with respect to μ and the relative entropy $H(\theta|\mu)$ is equal to $\mathbb{E}_\mu \left[\frac{d\theta}{d\mu} \log \frac{d\theta}{d\mu} \right]$. In [1], Boué and Dupuis use it to derive the variational formulation

$$(1.2) \quad -\log \mathbb{E}_\mu [e^{-f}] = \inf_u \mathbb{E}_\mu \left[f \circ (W + u) + \frac{1}{2} \int_0^1 |\dot{u}(s)|^2 ds \right]$$

where the infimum is taken over L^2 functions from \mathbb{W} to H whose density is adapted to (\mathcal{F}_t) . This variational formulation is useful to derive large deviation asymptotics as Laplace principles for small noise diffusions for instance. This result was later extended by Budhiraja and Dupuis to Hilbert-space-valued Brownian motions in [2], and then by Zhang to abstract Wiener spaces in [19], using the framework developed by Üstünel and Zakai in [16].

The Prékopa-Leindler theorem first formulation was given by Prékopa in [12] and arose in stochastic programming where a lot of non-linear optimization problems require concavity. In [8], Huu Hariya uses the variational formulation to retrieve a Prékopa-Leindler theorem for Wiener space, similar to the formulation of Üstünel and Feyel in [7] with log-concave measures. Other functional inequalities can be derived from 1.2, see for instance Lehec in [10].

The bounded from above hypothesis in 1.2 was weakened significantly by Üstünel in [18], it was replaced with the condition

$$\mathbb{E}_\mu [f e^{-f}] < \infty$$

and the existence of conjugate integers p and q such that

$$f \in L^p(\mu), e^{-f} \in L^q(\mu)$$

These relaxed hypothesis pave the way to new applications. The possibility of using unbounded functions is primordial in Dabrowski's application of 1.2 to free entropy in [4].

Üstünel's approach is routed in the study of the perturbations of the identity of \mathbb{W} , which is the coordinate process, and their invertibility. The question of the invertibility of an adapted perturbation of the identity is linked to the representability of measures and was put to light by the celebrated example of Tsirelson [15]. Üstünel proved that if $u \in L^2(\mu, H)$ has an adapted density, $I_{\mathbb{W}} + u$ is μ -a.s. invertible if and only if

$$H((I_{\mathbb{W}} + u)\mu|\mu) = \frac{1}{2} \mathbb{E}_\mu [|u|_H^2]$$

If u satisfies the hypothesis of Girsanov theorem, this gives a criteria of existence of strong solutions to some stochastic differential equations. Indeed, Üstünel proves in [17] that if such a $I_{\mathbb{W}} + u$ is invertible, its inverse V is a strong solution to the stochastic differential equation

$$dV(t) = -\dot{u}(t) \circ V dt + dW(t)$$

To prove 1.2 with the integrability conditions specified above, Üstünel uses the fact that H - C^1 shifts, meaning shifts that are a.s. Fréchet-differentiable on H with a a.s. continuous on H Fréchet derivative, are a.s. invertible, and that shifts can be approached with H - C^1 shifts using the Ornstein-Uhlenbeck semigroup.

In [9] we give a variational formulation similar as 1.2 for diffusions which are solutions of stochastic differential equations, while lowering the integrability hypothesis on f . This paper also focus on the invertibility of certain perturbations of the identity that are not affine shifts since the measure considered is not the same. However, contrary to what Üstünel does in [18], we do not approach f to derive invertible shift from those approached functions. We write $\frac{e^{-f}}{\mathbb{E}[e^{-f}]}$ as the Wick exponential of some v , and then approach v to obtain shifts that generate invertible perturbations of the identity. Our method relies on the fact that we have a Girsanov-like change of variable formula with the perturbations of the identity, with respect to a particular Brownian motion. It does not use any tool that is specific to Gaussian measures.

Two questions arise from this: can this method be applied to other measures, and can invertibility results be linked to the existence of strong solutions for some stochastic differential equations?

This paper presents a general framework to be able to similarly derive a variational formulation for $-\log \mathbb{E}_\nu[e^{-f}]$ for a large class of measures ν . We give a set of conditions so that a set of processes (W^u) can act as perturbations of W and allow a Girsanov-like change of variable with respect to a Brownian motion β . At first we want to have a minimal setting to be able to compute the variational formula, and we just consider the u whose density is a.s. bounded and we prove that

$$-\log \mathbb{E}_\nu [e^{-f}] = \inf_u \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right]$$

where the infimum is just taken over the u with a.s. bounded density, thus providing a clearer description of the infimum. The integrability hypothesis over f remain the same as in the case of a diffusion. In a second time, we study the possibilities to expand the domain over which the infimum is taken, for both variational calculus results, mainly concerning the attainability of the infimum, and invertibility results. Indeed, we prove that once again, W^u is invertible if and only if

$$H(W^u \nu | \nu) = \frac{1}{2} \mathbb{E}_\nu [|u|_H^2]$$

and in case of invertibility its inverse is of the form W^v . Furthermore, the invertibility of W^u can be related to the existence of strong solutions of stochastic differential equations in certain cases. If W^u can be written $W + w^u$, with $w^u \in L^0(\nu, H)$ having an adapted density, and is invertible, its inverse W^v verify

$$dW^v(t) = - \overbrace{w^u}^\cdot(t) \circ W^v dt + W(t)$$

We also prove a Prékopa-Leindler theorem on \mathbb{W} for the measure ν , however the convexity hypothesis seem very restrictive.

We apply this framework to various examples. First we retrieve the case of the image measure of a diffusion of [9], and then we study the case of the image measure of a Brownian bridge, a loop measure, and finally the image measure of a set of diffusing particles. The behaving of diffusing particles satisfying a differential stochastic system was studied by Cépa and Lepingle in [3], and Rogers and Shi in [14]. We focus on the case where the stochastic differential system the particles

(Z_1, \dots, Z_n) verify is of the form

$$\begin{aligned} Z_1(t) &= z_1(0) + \sigma B_1(t) + b \int_0^t Z_1(s) ds + ct + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{1\}} \int_0^t \frac{ds}{Z_1(s) - Z_j(s)} \\ &\vdots \\ Z_n(t) &= z_n(0) + \sigma B_n(t) + b \int_0^t Z_n(s) ds + ct + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{n\}} \int_0^t \frac{ds}{Z_n(s) - Z_j(s)} \end{aligned}$$

where (B_1, \dots, B_n) is a \mathbb{R}^n -valued Brownian motion and $\sigma^2 \leq 2\gamma$, which guarantee there is no collision.

2. Framework

Set $n \in \mathbb{N}^*$, we denote $\mathbb{W} = C([0, 1], \mathbb{R}^n)$ the canonical Wiener space, $H = \left\{ \int_0^\cdot \dot{h}(s) ds, \dot{h} \in L^2([0, 1]) \right\}$ the associated Cameron-Martin space and $(W(t))$ is the coordinate process.

We assume that \mathbb{W} is equipped with a probability measure ν . The filtration of a process m will be denoted (\mathcal{F}_t^m) , the filtration of W will be simply denoted (\mathcal{F}_t) . Except if stated otherwise, every filtration considered is completed with respect to ν . We denote, for $p \in \mathbb{R}_+$,

$$L_a^p(\nu, H) = \{u \in L^p(\nu, H), u \text{ is } (\mathcal{F}_t) - \text{adapted}\}$$

and

$$\mathcal{D} = \{u \in L_a^0(\nu, H), u \text{ is } d\nu \times dt - \text{a.s. bounded}\}$$

If m is a martingale and v has a density whose stochastic integral with respect to m is well defined we will denote

$$\delta_m v = \int_0^1 \dot{v}(s) dm(s)$$

We also denote the Wick exponential as follow

$$\rho(\delta_m v) = \exp \left(\int_0^1 \dot{v}(s) dm(s) - \frac{1}{2} \int_0^1 |\dot{v}(s)|^2 d\langle m \rangle(s) \right)$$

and for $p \geq 0$ we denote

$$G_p(\nu, m) = \{u \in L_a^p(\nu, H), \mathbb{E}_\nu [\rho(-\delta_m u)] = 1\}$$

We assume there exists a family of processes $(W^u)_{u \in \mathcal{D}}$ and a ν -Brownian motion β which verify the following conditions:

- (i) β is a ν -Brownian motion whose canonical filtration is identical to the canonical filtration of $(W(t))$
- (ii) $W^0 = W$
- (iii) For every $u \in \mathcal{D}$, the law of W^u under $\tilde{\nu}^u$ is the same as the law of W under ν , where $\tilde{\nu}^u$ is defined by $\frac{d\tilde{\nu}^u}{d\nu} = \rho(-\delta_\beta u)$
- (iv) For every $u \in \mathcal{D}$,

$$\beta \circ W^u = \beta + u$$

- (v) For every $u, v \in \mathcal{D}$,

$$W^u \circ W^v = W^{v+u \circ W^v} \quad \nu - \text{a.s.}$$

Remark: Clearly $\mathcal{D} \subset L_a^\infty(\nu, H)$, so if $u \in \mathcal{D}$, $\mathbb{E}_\nu[\rho(-\delta_\beta u)] = 1$ and $\tilde{\nu}^u$ which was defined in condition (iii) is indeed a probability measure.

Condition (iii) can be written as follow:

Proposition 1. *Set $u \in \mathcal{D}$, for every bounded measurable function f , we have:*

$$\mathbb{E}_\nu[f] = \mathbb{E}_\nu[f \circ W^u \rho(-\delta_\beta u)]$$

Next proposition ensures that the compositions written in (iv) and (v) are well defined.

Proposition 2. *Set $u \in \mathcal{D}$, we have*

$$W^u \nu \sim \nu$$

Proof: Set $f \in C_b(\mathbb{W})$ bounded and measurable, we have, using proposition 1

$$\begin{aligned} \mathbb{E}_{W^u \tilde{\nu}^u}[f] &= \mathbb{E}_{\tilde{\nu}^u}[f \circ W^u] \\ &= \mathbb{E}_\nu[f \circ W^u \rho(-\delta_\beta u)] \\ &= \mathbb{E}_\nu[f] \end{aligned}$$

so $W^u \tilde{\nu}^u = \nu$.

Since $\tilde{\nu}^u \sim \nu$, we have $W^u \tilde{\nu}^u \sim W^u \nu$ which conclude the proof. \square

3. Invertibility results

3.1. First results.

Definition 1. *A measurable map $U : \mathbb{W} \rightarrow \mathbb{W}$ is said to be ν -a.s. left-invertible if and only if $U\nu \ll \nu$ and there exists a measurable map $V : \mathbb{W} \rightarrow \mathbb{W}$ such that $V \circ U = I_{\mathbb{W}}$ ν -a.s.*

A measurable map $U : \mathbb{W} \rightarrow \mathbb{W}$ is said to be ν -a.s. right-invertible if and only if there exists a measurable map $V : \mathbb{W} \rightarrow \mathbb{W}$ such that $V\nu \ll \nu$ and $U \circ V = I_{\mathbb{W}}$ ν -a.s.

Proposition 3. *Set $U, V : \mathbb{W} \rightarrow \mathbb{W}$ measurable maps such that $V \circ U = I_{\mathbb{W}}$ ν -a.s. and $V\nu \ll \nu$. Then $U \circ V = I_{\mathbb{W}}$ $U\nu$ -a.s., so if $U\nu \sim \nu$, we also have $U \circ V = I_{\mathbb{W}}$ ν -a.s. In that case, we will say that U is ν -a.s. invertible and we also have $V\nu \sim \nu$.*

Proof: There exists $A \subset W$ such that $\nu(A) = 1$ and for every $w \in A$, $V \circ U(w) = w$. Consider such a set A , we have

$$\begin{aligned} \mathbb{E}_{U\nu}[1_{U \circ V(w)=w}] &= \mathbb{E}_\nu[1_{U \circ V \circ U(w)=U(w)}] \\ &= \mathbb{E}_\nu[1_{U \circ V \circ U(w)=U(w)} 1_{w \in A}] + \mathbb{E}_\nu[1_{U \circ V \circ U(w)=U(w)} 1_{w \notin A}] \\ &= \mathbb{E}_\nu[1_{U(w)=U(w)} 1_{w \in A}] \\ &= 1 \end{aligned}$$

Now assume that U is ν -a.s. invertible, set $A \in \mathcal{F}_1$ such that $V\nu(A) = 0$. We have $1_A \circ V = 0$ ν -a.s. and since $U\nu \sim \nu$, we have $1_A \circ V = 0$ $U\nu$ -a.s. Finally,

$$\nu(A) = \mathbb{E}_\nu[1_A] = \mathbb{E}_\nu[1_A \circ V \circ U]$$

which concludes the proof. \square

Theorem 1. *Set $u \in \mathcal{D}$ and assume there exists a measurable map $V : \mathbb{W} \rightarrow \mathbb{W}$ such that $V \circ W^u = I_{\mathbb{W}}$ ν -a.s. Denote $v = -u \circ V$. Then $W^u \circ V = I_{\mathbb{W}}$ ν -a.s., $v \in \mathcal{D}$ and $V = W^v$ ν -a.s. Moreover, we have*

$$\begin{aligned} \frac{dW^u \nu}{d\nu} &= \rho(-\delta_\beta v) \\ \frac{dW^v \nu}{d\nu} &= \rho(-\delta_\beta u) \end{aligned}$$

Proof: Set $f \in C_b(\mathbb{W})$, we have

$$\mathbb{E}_\nu [f \circ V] = \mathbb{E}_\nu [f \circ V \circ W^u \rho(-\delta_\beta u)] = \mathbb{E}_\nu [f \rho(-\delta_\beta u)]$$

So $V\nu \sim \nu$ and

$$\frac{dV\nu}{d\nu} = \rho(-\delta_\beta u)$$

Since $W^u \nu \sim \nu$, the first assertion comes from proposition 3.

Clearly $v \in L^0(\nu, H)$. Since $u \in \mathcal{D}$, there exists $n \in \mathbb{N}$ such that $d\nu \times dt$ -a.s.

$$|\dot{u}(s, w)| \leq n$$

Since $V\nu \ll \nu$, we have $d\nu \times dt$ -a.s.

$$|\dot{v}(s, w)| \leq n$$

Finally, let us prove that \dot{v} is adapted. We have ν -a.s.

$$\dot{v} \circ W^u = -\dot{u} \circ V \circ W^u = -\dot{u}$$

hence $v \circ W^u$ is adapted. Set $A \in L^2(d\nu \times dt)$ an adapted process. We have:

$$\begin{aligned} \mathbb{E}_\nu \left[\rho(-\delta_\beta u) \int_0^1 \dot{v}(s) \circ W^u A(s) \circ W^u ds \right] &= \mathbb{E}_\nu \left[\int_0^1 \dot{v}(s) A(s) ds \right] \\ &= \mathbb{E}_\nu \left[\int_0^1 \mathbb{E}_\nu [\dot{v}(s) | \mathcal{F}_s] A(s) ds \right] \\ &= \mathbb{E}_\nu \left[\rho(-\delta_\beta u) \int_0^1 \mathbb{E}_\nu [\dot{v}(s) | \mathcal{F}_s] \circ W^u A(s) \circ W^u ds \right] \end{aligned}$$

So $\mathbb{E}_\nu [\dot{v}(s) | \mathcal{F}_s] \circ W^u = \dot{v}(s) \circ W^u$ $ds \times d\nu$ -a.s. which implies $\mathbb{E}_\nu [\dot{v}(s) | \mathcal{F}_s] = \dot{v}(s)$ $ds \times d\nu$ -a.s. since $W^u \nu \sim \nu$.

We have

$$W^v \circ W^u = W^{u+v \circ W^u} = W^0 = I_{\mathbb{W}} \quad \nu - a.s.$$

and

$$V = W^v \circ W^u \circ V = W^v \quad \nu - a.s.$$

Finally, set $f \in C_b(\mathbb{W})$,

$$\begin{aligned} \mathbb{E}_\nu [f \circ W^u] &= \mathbb{E}_\nu [f \circ W^u \circ W^v \rho(-\delta_\beta v)] \\ &= \mathbb{E}_\nu [f \rho(-\delta_\beta v)] \\ \mathbb{E}_\nu [f \circ W^v] &= \mathbb{E}_\nu [f \circ W^v \circ W^u \rho(-\delta_\beta u)] \\ &= \mathbb{E}_\nu [f \rho(-\delta_\beta u)] \end{aligned}$$

which gives the final assertion. \square

Remark: Set $u \in \mathcal{D}$, W^u is ν -a.s. invertible if and only if it is ν -a.s. left-invertible.

3.2. Entropic characterisation of the invertibility of W^u . In this section, we prove that the process W^u is left invertible if and only if the kinetic energy of the perturbation u is equal to the relative entropy of $W^u \nu$.

Proposition 4. *Set $u \in \mathcal{D}$. We have:*

$$H(W^u \nu | \nu) \leq \frac{1}{2} \mathbb{E}_\nu [|u|_H^2]$$

Proof: Set $g \in C_b(\mathbb{W})$ and denote $L = \frac{dW^u \nu}{d\nu}$, we have:

$$\begin{aligned} \mathbb{E}_\nu [g \circ W^u] &= \mathbb{E}_\nu [gL] \\ &= \mathbb{E}_\nu [g \circ W^u L \circ W^u \rho(-\delta_\beta u)] \end{aligned}$$

Hence $L \circ W^u \mathbb{E}_\nu [\rho(-\delta_\beta u) | \mathcal{F}_1^{W^u}] = 1$ ν -a.s. and

$$\begin{aligned} H(W^u \nu | \nu) &= \mathbb{E}_\nu [L \log L] \\ &= \mathbb{E}_{W^u \nu} [\log L] \\ &= \mathbb{E}_\nu [\log L \circ W^u] \\ &= -\mathbb{E}_\nu \left[\log \mathbb{E}_\nu \left[\rho(-\delta_\beta u) | \mathcal{F}_1^{W^u} \right] \right] \\ &\leq -\mathbb{E}_\nu [\log \rho(-\delta_\beta u)] \\ &\leq \frac{1}{2} \mathbb{E}_\nu [|u|_H^2] \end{aligned}$$

□

The proof gives the following additional result

Corollary 1. *For $u \in \mathcal{D}$, we have*

$$L \circ W^u \mathbb{E}_\nu \left[\rho(-\delta_\beta u) | \mathcal{F}_1^{W^u} \right] = 1$$

Now comes the criteria:

Theorem 2. *Set $u \in \mathcal{D}$, then W^u is ν -a.s. invertible if and only if:*

$$H(W^u \nu | \nu) = \frac{1}{2} \mathbb{E}_\nu [|u|_H^2]$$

Proof: Assume that the inequality hold. We still denote $L = \frac{dW^u \nu}{d\nu}$ and as in last proof we have ν -a.s.

$$L \circ W^u \mathbb{E}_\nu \left[\rho(-\delta_\beta u) | \mathcal{F}_1^{W^u} \right] = 1$$

Using Jensen inequality we have ν -a.s.

$$\begin{aligned} 0 &= \log L \circ W^u + \log \mathbb{E}_\nu \left[\rho(-\delta_\beta u) | \mathcal{F}_1^{W^u} \right] \\ &\geq \log L \circ W^u + \mathbb{E}_\nu \left[\log \rho(-\delta_\beta u) | \mathcal{F}_1^{W^u} \right] \end{aligned}$$

and

$$\begin{aligned}
0 &\geq \mathbb{E}_\nu [\log L \circ W^u] + \mathbb{E}_\nu [\log \rho(-\delta_\beta u)] \\
&\geq H(W^u | \nu) - \frac{1}{2} \mathbb{E}_\nu [|u|_H^2] \\
&= 0
\end{aligned}$$

So

$$\begin{aligned}
0 &= \log L \circ W^u + \log \mathbb{E}_\nu [\rho(-\delta_\beta u) | \mathcal{F}_1^{W^u}] \\
&= \log L \circ W^u + \mathbb{E}_\nu [\log \rho(-\delta_\beta u) | \mathcal{F}_1^{W^u}]
\end{aligned}$$

and

$$\log \mathbb{E}_\nu [\rho(-\delta_\beta u) | \mathcal{F}_1^{W^u}] = \mathbb{E}_\nu [\log \rho(-\delta_\beta u) | \mathcal{F}_1^{W^u}]$$

The strict concavity of the function \log gives

$$\mathbb{E}_\nu [\rho(-\delta_\beta u) | \mathcal{F}_1^{W^u}] = \rho(-\delta_\beta u)$$

Finally we have

$$(3.3) \quad L \circ W^u \rho(-\delta_\beta u) = 1$$

Since β is a ν -Brownian motion, there exists $v \in L_a^0(\nu, H)$ such that $L = \rho(-\delta_\beta v)$.

We apply the logarithm to 3.3 to get:

$$0 = \delta_\beta v \circ W^u + \frac{1}{2} |v \circ W^u|_H + \delta_\beta u + \frac{1}{2} |u|_H$$

We have:

$$\delta_\beta v \circ W^u = \int_0^1 \dot{v}(s) \circ W^u d\beta(s) + \langle v \circ W^u, u \rangle_H$$

so finally we have:

$$(3.4) \quad 0 = \delta_\beta (v \circ W^u + u) + \frac{1}{2} |v \circ W^u + u|_H^2$$

According to Girsanov theorem $\beta + v$ is a $W^u \nu$ -Brownian motion, so:

$$\begin{aligned}
\mathbb{E}_\nu [L \log L] &= \mathbb{E}_{W^u \nu} [\log L] \\
&= \mathbb{E}_{W^u \nu} \left[- \int_0^1 \dot{v}(s) d\beta(s) - \frac{1}{2} \int_0^1 |\dot{v}(s)|^2 ds \right] \\
&= \frac{1}{2} \mathbb{E}_{W^u \nu} \left[\int_0^1 |\dot{v}(s)|^2 ds \right] \\
&= \frac{1}{2} \mathbb{E}_\nu [|v \circ W^u|_H^2]
\end{aligned}$$

So $v \circ W^u \in L_a^2(\nu, H)$ and we can take the expectation with respect to ν in 3.4 to obtain $u + v \circ W^u = 0$ ν -a.s. which implies $v \in \mathcal{D}$. So ν -a.s.

$$W^v \circ W^u = W^{u+v \circ W^u} = W^0 = I_{\mathbb{W}}$$

Conversely, assume that W^u admits an inverse V and set $v = -u \circ V$. According to theorem 1, $v \in D$ and $W^v = V$ ν -a.s. Once again, denote $L = \frac{dW^u \nu}{d\nu}$. We know that $L = \rho(-\delta_\beta v)$. Observe that

$$\begin{aligned} \log L \circ W^u &= \left(-\int_0^1 \dot{v}(s) d\beta(s) - \frac{1}{2} \int_0^1 \dot{v}(s)^2 ds \right) \circ W^u \\ &= -\log \rho(-\delta_\beta u) \end{aligned}$$

So finally

$$\begin{aligned} H(W^u \nu | \nu) &= \mathbb{E}_\nu [L \log L] = \mathbb{E}_\nu [\log L \circ W^u] \\ &= \mathbb{E}_\nu [-\log \rho(-\delta_\beta u)] \\ &= \frac{1}{2} \mathbb{E}_\nu [|u|_H^2] \end{aligned}$$

□

The proof gives the following additional result:

Corollary 2. *Set $u \in \mathcal{D}$ such that W^u is ν -a.s. left-invertible, we have*

$$L \circ W^u \rho(-\delta_\beta u) = 1$$

The following corollary is an immediate consequence of theorems 1 and 2.

Corollary 3. *Set $u \in \mathcal{D}$, we have*

$$H(W^u \nu | \nu) = \frac{1}{2} \mathbb{E}_\nu [|u|_H^2]$$

if and only if there exists $v \in \mathcal{D}$ such that

$$W^v \circ W^u = W^u \circ W^v = I_{\mathbb{W}} \nu - a.s.$$

Definition 2. *We define \mathcal{D}^i as*

$$\mathcal{D}^i = \{u \in D, W^u \text{ is } \nu - a.s. \text{ invertible}\}$$

4. Variational problem

4.1. Approximation of absolutely continuous measures.

Theorem 3. *If $\theta \sim \nu$ is such that there exists $r > 1$ such that*

$$\frac{d\theta}{d\nu} \log \frac{d\theta}{d\nu} \in L^1(\nu)$$

and

$$\log \frac{d\theta}{d\nu} \in L^r(\nu)$$

there exists $(u_n) \in (\mathcal{D}^i)^\mathbb{N}$ such that,

$$\begin{aligned} \frac{dW^{u_n} \nu}{d\nu} \log \frac{dW^{u_n} \nu}{d\nu} &\rightarrow \frac{d\theta}{d\nu} \log \frac{d\theta}{d\nu} \text{ in } L^1(\nu) \\ \frac{dW^{u_n} \nu}{d\nu} \log \frac{d\theta}{d\nu} &\rightarrow \frac{d\theta}{d\nu} \log \frac{d\theta}{d\nu} \text{ in } L^1(\nu) \end{aligned}$$

Proof: Denote

$$L = \frac{d\theta}{d\nu}$$

Eventually sequentializing afterward, we have to prove that for any $\epsilon > 0$, there exists $u \in \mathcal{D}^i$ such that

$$\begin{aligned} \mathbb{E}_\nu \left[\left| \frac{dW^u_\nu}{d\nu} \log \frac{dW^u_\nu}{d\nu} - L_1 \log L_1 \right| \right] &\leq \epsilon \\ \mathbb{E}_\nu \left[\left| \frac{dW^u_\nu}{d\nu} \log L_1 - L_1 \log L_1 \right| \right] &\leq \epsilon \end{aligned}$$

The proof is divided in six steps.

Step 1 : We approximate L with a density that is both lower-bounded and upper bounded.

Denote

$$\begin{aligned} \phi_n &= \min(L, n) \\ L_n &= \frac{\phi_n}{\mathbb{E}_\nu[\phi_n]} \end{aligned}$$

The monotone convergence theorem ensures that $\mathbb{E}_\nu[\phi_n] \rightarrow 1$ so for any $\alpha \in (0, 1)$, there exists some $n_\alpha \in \mathbb{N}$ such that for any $n \geq n_\alpha$,

$$\mathbb{E}_\nu[\phi_n] \geq \alpha$$

$(L_n \log L_n)$ converges ν -a.s. to $L \log L$ and if $n \geq n_\alpha$ and

$$\begin{aligned} |L_n \log L_n| &= \left| \frac{\phi_n}{\mathbb{E}_\nu[\phi_n]} \log \frac{\phi_n}{\mathbb{E}_\nu[\phi_n]} \right| 1_{\frac{\phi_n}{\mathbb{E}_\nu[\phi_n]} \leq 1} + \left| \frac{\phi_n}{\mathbb{E}_\nu[\phi_n]} \log \frac{\phi_n}{\mathbb{E}_\nu[\phi_n]} \right| 1_{\frac{\phi_n}{\mathbb{E}_\nu[\phi_n]} > 1} \\ &\leq e^{-1} 1_{\frac{\phi_n}{\mathbb{E}_\nu[\phi_n]} \leq 1} + \left| \frac{L}{\alpha} \log \frac{L}{\alpha} \right| 1_{\frac{\phi_n}{\mathbb{E}_\nu[\phi_n]} > 1} \\ &\leq e^{-1} + \left| \frac{L}{\alpha} \log \frac{L}{\alpha} \right| \end{aligned}$$

So the Lebesgue theorem ensures that $(L_n \log L_n)$ converge toward $L \log L$ in $L^1(\nu)$. Similarly, $(L_n \log L)$ converges ν -a.s. to $L \log L$ and if $n \leq n_\alpha$,

$$|L_n \log L| \leq \left| \frac{L}{\alpha} \log L \right|$$

and the Lebesgue theorem ensures that $(L_n \log L)$ converges to $L_n \log L$ in $L^1(\nu)$, so there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \mathbb{E}_\nu[|L_{n_0} \log L_{n_0} - L \log L|] &\leq \epsilon \\ \mathbb{E}_\nu[|L_{n_0} \log L - L \log L|] &\leq \epsilon \end{aligned}$$

$\left(\frac{L_{n_0} + a}{1 + a} \log \frac{L_{n_0} + a}{1 + a} \right)$ converges ν -a.s. to $L_{n_0} \log L_{n_0}$ when a converges to 0. Set $a \in [0, 1]$, we have

$$\begin{aligned} \left| \frac{L_{n_0} + a}{1 + a} \log \frac{L_{n_0} + a}{1 + a} \right| &= \left| \frac{L_{n_0} + a}{1 + a} \log \frac{L_{n_0} + a}{1 + a} \right| 1_{L_{n_0} \leq 1} + \left| \frac{L_{n_0} + a}{1 + a} \log \frac{L_{n_0} + a}{1 + a} \right| 1_{L_{n_0} > 1} \\ &\leq e^{-1} 1_{L_{n_0} \leq 1} + |L_{n_0} \log L_{n_0}| 1_{L_{n_0} > 1} \\ &\leq e^{-1} + |L_{n_0} \log L_{n_0}| \end{aligned}$$

So the Lebesgue theorem ensures that $\left(\frac{L_{n_0}+a}{1+a} \log \frac{L_{n_0}+a}{1+a}\right)$ converges to $L_{n_0} \log L_{n_0}$ in $L^1(\nu)$. Similarly, $\left(\frac{L_{n_0}+a}{1+a} \log L\right)$ converges ν -a.s. to $L_{n_0} \log L$ and

$$\left| \frac{L_{n_0}+a}{1+a} \log L \right| \leq |(L_{n_0}+1) \log L|$$

and the Lebesgue theorem ensures that $\left(\frac{L_{n_0}+a}{1+a} \log L\right)$ converges to $L_{n_0} \log L$ in $L^1(\nu)$ and there exists $a \in [0, 1]$ such that

$$\begin{aligned} \mathbb{E}_\nu \left[\left| \frac{L_{n_0}+a}{1+a} \log \frac{L_{n_0}+a}{1+a} - L_{n_0} \log L_{n_0} \right| \right] &\leq \epsilon \\ \mathbb{E}_\nu \left[\left| \frac{L_{n_0}+a}{1+a} \log L - L_{n_0} \log L \right| \right] &\leq \epsilon \end{aligned}$$

$\frac{L_{n_0}+a}{1+a}$ is both lower-bounded and upper-bounded in $L^\infty(\nu)$, denote these bounds respectively d and D .

Also denote

$$M(t) = \mathbb{E}_\nu \left[\frac{L_{n_0}+a}{1+a} \middle| \mathcal{F}_t \right]$$

We write

$$M(t) = \exp \left(\int_0^t \dot{\alpha}(s) d\beta(s) - \frac{1}{2} \int_0^t |\dot{\alpha}(s)|^2 ds \right)$$

with $\alpha \in L_a^0(\nu, H)$.

Step 2 : We prove that $\alpha \in L^2(\nu, H)$.

Set

$$T_n = \inf \left\{ t \in [0, 1], \int_0^t |\dot{\alpha}(s)|^2 ds > n \right\}$$

(T_n) is a sequence of stopping times which increases stationarily toward 1. We have, using $M = 1 + \int_0^\cdot \dot{\alpha}(s) M(s) d\beta(s)$

$$\begin{aligned} \mathbb{E}_\nu \left[(M(t \wedge T_n) - 1)^2 \right] &= \mathbb{E}_\nu \left[\int_0^{t \wedge T_n} |\dot{\alpha}(s)|^2 M(s)^2 ds \right] \\ &\geq d^2 \mathbb{E}_\nu \left[\int_0^{t \wedge T_n} |\dot{\alpha}(s)|^2 ds \right] \end{aligned}$$

so

$$\mathbb{E}_\nu \left[\int_0^{t \wedge T_n} |\dot{\alpha}(s)|^2 ds \right] \leq \frac{1}{d^2} \mathbb{E}_\nu \left[(M(t \wedge T_n) - 1)^2 \right] \leq \frac{2(D^2 + 1)}{d^2}$$

hence passing to the limit

$$\mathbb{E}_\nu \left[\int_0^1 |\dot{\alpha}(s)|^2 ds \right] \leq \infty$$

Step 3 : we approximate α with an element of $L^\infty(\nu, H)$.

Define

$$\alpha_n(t, w) \in \mathbb{R} \times W \mapsto \int_0^t \dot{\alpha}(s, w) 1_{[0, T_n]}(s, w) ds$$

and

$$M^n(t) = \exp \left(\int_0^t \dot{\alpha}^n(s) d\beta(s) - \frac{1}{2} \int_0^t |\dot{\alpha}^n(s)|^2 ds \right)$$

and clearly $(M^n(1) \log M^n(1))$ converges ν -a.s. to $M(1) \log M(1)$, $(M^n(1) \log L)$ converges ν -a.s. to $M(1) \log L$ and $M^n(1) = \mathbb{E}_\nu [M(1) | \mathcal{F}_{T_n}]$, so ν -a.s.

$$\begin{aligned} |M^n(1) \log M^n(1)| &\leq \max(e^{-1}, |D \log D|) \\ |M^n(1) \log L| &\leq |D \log L| \end{aligned}$$

so the Lebesgue theorem ensures that $(M^n(1) \log M^n(1))$ converges to $M(1) \log M(1)$ in $L^1(\nu)$ and $(M^n(1) \log L)$ converges to $M(1) \log L$ in $L^1(\nu)$ and there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} |M^n(1) \log M^n(1) - M(1) \log M(1)| &\leq \epsilon \\ |M^n(1) \log L - M(1) \log L| &\leq \epsilon \end{aligned}$$

Step 4 : We approximate α^n with an element of \mathcal{D} .

Define

$$\xi^{n,m} : (t, w) \in [0, 1] \times \mathbb{W} \mapsto \int_0^t \max(\min(\dot{\alpha}^n(s, w), m), -m) ds$$

and

$$M^{n,m}(t) = \exp \left(\int_0^t \xi^{n,m}(s) d\beta(s) - \frac{1}{2} \int_0^t |\xi^{n,m}(s)|^2 ds \right)$$

$(M^{n,m}(1) \log M^{n,m}(1))$ and $(M^{n,m}(1) \log L)$ converges respectively to $M^n(1) \log M^n(1)$ and $M^n(1) \log L$ in probability. To prove that $(M^{n,m}(1) \log M^{n,m}(1))$ is uniformly integrable, it is sufficient to prove it is bounded in any $L^p(\nu)$, set $p > 1$

$$\begin{aligned} \mathbb{E}_\nu [|M^{n,m}(1)|^p] &= \mathbb{E}_\nu \left[\exp \left(p \int_0^1 \xi^{n,m}(s) d\beta(s) - \frac{p}{2} \int_0^1 |\xi^{n,m}(s)|^2 ds \right) \right] \\ &= \mathbb{E}_\nu \left[\exp \left(p \int_0^1 \xi^{n,m}(s) d\beta(s) - \frac{p^2}{2} \int_0^1 |\xi^{n,m}(s)|^2 ds \right) \exp \left(\frac{p^2 - p}{2} \int_0^1 |\xi^{n,m}(s)|^2 ds \right) \right] \\ &\leq \mathbb{E}_\nu \left[\exp \left(\int_0^1 p \xi^{n,m}(s) d\beta(s) - \frac{1}{2} \int_0^1 |p \xi^{n,m}(s)|^2 ds \right) \exp \left(\frac{p^2 - p}{2} n \right) \right] \\ &\leq \exp \left(\frac{p^2 - p}{2} n \right) \end{aligned}$$

so $(M^{n,m}(1) \log M^{n,m}(1))$ converges to $M^n(1) \log M^n(1)$ in $L^1(\nu)$. Furthermore, set p such that $p^{-1} + r^{-1} = 1$

$$\begin{aligned} \mathbb{E}_\nu [|M^{n,m}(1) \log L - M^n(1) \log L|] &\leq |M^{n,m}(1) - M^n(1)|_{L^p(\nu)} |\log L|_{L^r(\nu)} \\ &\rightarrow 0 \end{aligned}$$

and there exists some $m > 0$ such that

$$\begin{aligned} \mathbb{E}_\nu [|M^{n,m}(1) \log M^{n,m}(1) - M^n(1) \log M^n(1)|] &\leq \epsilon \\ \mathbb{E}_\nu [|M^{n,m}(1) \log L - M^n(1) \log L|] &\leq \epsilon \end{aligned}$$

Step 5 : we approximate $\xi^{n,m}$ with a retarded shift.

For $\eta > 0$ set

$$\begin{aligned} \dot{\gamma}^\eta(t, w) &\in [0, 1] \times W \mapsto \xi^{n,m}(t - \eta)(w) 1_{t > \eta} \\ N^\eta(t) &= \exp \left(\int_0^1 \dot{\gamma}^\eta(s) d\beta(s) - \frac{1}{2} \int_0^1 |\dot{\gamma}^\eta(s)|^2 ds \right) \end{aligned}$$

We have for any $\eta > 0$, $d\nu \times ds$ -a.s.,

$$|\dot{\gamma}^\eta(s)| \leq m$$

i.e. $\gamma^\eta \in \mathcal{D}$.

Similarly as before $(N^\eta(1) \log N^\eta(1))$ converges in probability to $M^{n,m}(1) \log M^{n,m}(1)$ and $(N^\eta(1))$ is bounded in every $L^p(\nu)$ hence $(N^\eta(1) \log N^\eta(1))$ is uniformly integrable and converges in $L^1(\nu)$ to $M^{n,m}(1) \log M^{n,m}(1)$

Furthermore, using Holder inequality, we have

$$\mathbb{E}_\nu [|N^\eta(1) \log L - M^{n,m}(1) \log L|] \leq |N^\eta(1) - M^{n,m}(1)|_{L^p(\nu)} |\log L|_{L^r(\nu)}$$

where $\frac{1}{p} + \frac{1}{r} = 1$.

Consequently there exists $\eta > 0$ such that

$$\begin{aligned} \mathbb{E}_\nu [|N^\eta(1) \log N^\eta(1) - M^{n,m}(1) \log M^{n,m}(1)|] &\leq \epsilon \\ \mathbb{E}_\nu [|N^\eta(1) \log L - M^{n,m}(1) \log L|] &\leq \epsilon \end{aligned}$$

using triangular inequality, we have

$$\begin{aligned} \mathbb{E}_\nu [|L \log L - N^\eta(1) \log N^\eta(1)|] &\leq \mathbb{E}_\nu [|L \log L - L_{n_0} \log L_{n_0}|] \\ &\quad + \mathbb{E}_\nu \left[\left| L_{n_0} \log L_{n_0} - \frac{L_{n_0} + a}{1 + a} \log \frac{L_{n_0} + a}{1 + a} \right| \right] \\ &\quad + \mathbb{E}_\nu \left[\left| \frac{L_{n_0} + a}{1 + a} \log \frac{L_{n_0} + a}{1 + a} - M^n(1) \log M^n(1) \right| \right] \\ &\quad + \mathbb{E}_\nu [|M^n(1) \log M^n(1) - M^{n,m}(1) \log M^{n,m}(1)|] \\ &\quad + \mathbb{E}_\nu [|M^{n,m}(1) \log M^{n,m}(1) - N^\eta(1) \log N^\eta(1)|] \\ &\leq 5\epsilon \\ \mathbb{E}_\nu [|L \log L - N^\eta(1) \log L|] &\leq \mathbb{E}_\nu [|L \log L - L_{n_0} \log L|] \\ &\quad + \mathbb{E}_\nu \left[\left| L_{n_0} \log L - \frac{L_{n_0} + a}{1 + a} \log L \right| \right] \\ &\quad + \mathbb{E}_\nu \left[\left| \frac{L_{n_0} + a}{1 + a} \log L - M^n(1) \log L \right| \right] \\ &\quad + \mathbb{E}_\nu [|M^n(1) \log L - M^{n,m}(1) \log L|] \\ &\quad + \mathbb{E}_\nu [|M^{n,m}(1) \log L - N^\eta(1) \log L|] \\ &\leq 5\epsilon \end{aligned}$$

Step 6 : We prove that $W^{-\gamma^\eta}$ is ν -a.s. left-invertible and is the solution to our problem.

Set $A \subset \mathbb{W}$ such that $\nu(A) = 1$ and for every $w \in A$, $\beta \circ W^{-\gamma^\eta}(w) = \beta(w) - \gamma^\eta(w)$ and set $w_1, w_2 \in A$ such that $W^{-\gamma^\eta}(w_1) = W^{-\gamma^\eta}(w_2)$. We have

$$\begin{aligned} \beta \circ W^{-\gamma^\eta}(w_1) &= \beta \circ W^{-\gamma^\eta}(w_2) \\ \beta(w_1) - \int_0^\cdot \dot{\gamma}^\eta(s, w_1) ds &= \beta(w_2) - \int_0^\cdot \dot{\gamma}^\eta(s, w_2) ds \end{aligned}$$

For any $s \in [0, \eta]$, $\beta(s, w_1) = \beta(s, w_2)$, γ^η being adapted to filtration $(\mathcal{F}_{s-\eta}^\beta)$, it implies that for $s \in [0, 2\eta]$

$$\int_0^s \dot{\gamma}^\eta(r, w_1) ds = \int_0^s \dot{\gamma}^\eta(r, w_2) ds$$

and

$$\beta(s, w_1) = \beta(s, w_2)$$

An easy iteration shows that $\beta(w_1) = \beta(w_2)$.

Since β and W have the same filtrations, and β is ν -a.s. path-continuous, we can write $W(t) = \phi_t(\beta(s), s \in [0, t] \cap \mathbb{Q})$ ν -a.s. for every $t \in [0, 1]$, with ϕ a measurable function from $\mathbb{R}^{\mathbb{Q}} \times \mathbb{Q}$ to \mathbb{R} , see [11]. Consequently, we can write $(W(t), t \in [0, 1] \cap \mathbb{Q}) = \phi(\beta(t), t \in [0, 1] \cap \mathbb{Q})$ ν -a.s., with ϕ a measurable function from $\mathbb{R}^{\mathbb{Q}}$ to $\mathbb{R}^{\mathbb{Q}}$. Denote

$$A' = A \cap \{w \in \mathbb{W}, (W(t, w), t \in [0, 1] \cap \mathbb{Q}) = \phi(\beta(t, w), t \in [0, 1] \cap \mathbb{Q})\}$$

$\nu(A') = 1$. Set $w_1, w_2 \in A'$ such that $W^{-\gamma^\eta}(w_1) = W^{-\gamma^\eta}(w_2)$. We have $\beta(w_1) = \beta(w_2)$ so

$$\begin{aligned} (W(t, w_1), t \in [0, 1] \cap \mathbb{Q}) &= (W(t, w_2), t \in [0, 1] \cap \mathbb{Q}) \\ (w_1(t), t \in [0, 1] \cap \mathbb{Q}) &= (w_2(t), t \in [0, 1] \cap \mathbb{Q}) \end{aligned}$$

w_1 and w_2 are continuous and coincide on $[0, 1] \cap \mathbb{Q}$ so they are equal.

$W^{-\gamma^\eta}$ is ν -a.s. injective and so ν -a.s. left-invertible, its inverse is of the form W^{v^η} , with $v^\eta \in \mathcal{D}$ and we have

$$\frac{dW^{v^\eta} \nu}{d\nu} = L_1^{\eta, n}$$

So $W^{v^\eta} \nu \sim \nu$ and

$$W^{v^\eta} \circ W^{-\gamma^\eta} = W^{-\gamma^\eta} \circ W^{v^\eta} \quad \nu - a.s.$$

□

4.2. Main theorem. As stated in the beginning, we aim to provide a variational representation of $-\log \mathbb{E}_\nu [e^{-f}]$. This first result is from [18]:

Theorem 4. *Set $f : \mathbb{W} \rightarrow \mathbb{R}$ a measurable function verifying*

$$\mathbb{E}_\nu [|f|(1 + e^{-f})] < \infty$$

Denote \mathcal{P} the set of probability measures on $(\mathbb{W}, \mathcal{F})$ which are absolutely continuous with respect to ν , then

$$-\log \mathbb{E}_\nu [e^{-f}] = \inf_{\theta \in \mathcal{P}} (\mathbb{E}_\theta[f] + H(\theta|\nu))$$

and the unique supremum is attained at the measure

$$d\theta_0 = \frac{e^{-f}}{\mathbb{E}_\nu [e^{-f}]} d\nu$$

Proposition 5. *Set $f : \mathbb{W} \rightarrow \mathbb{R}$ a measurable function verifying $\mathbb{E}_\nu [|f|(1 + e^{-f})] < \infty$, then*

$$-\log \mathbb{E}_\nu [e^{-f}] \leq \inf_{u \in \mathcal{D}} \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right]$$

Proof: Set $u \in \mathcal{D}$

$$\begin{aligned} -\log \mathbb{E}_\nu [e^{-f}] &\leq \mathbb{E}_{W^u \nu} [f] + H(W^u \nu | \nu) \\ &= \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right] \end{aligned}$$

□

Here is the main result.

Theorem 5. Set $p > 1$ and $f \in L^p(\nu)$ such that $\mathbb{E}_\nu [(|f| + 1)e^{-f}] < \infty$, then we have

$$-\log \mathbb{E}_\nu [e^{-f}] = \inf_{u \in \mathcal{D}^i} \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right]$$

Proof: Using proposition 5, we have easily

$$-\log \mathbb{E}_\nu [e^{-f}] \leq \inf_{u \in \mathcal{D}^i} \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right]$$

Let θ_0 be the measure on \mathbb{W} defined by

$$d\theta_0 = \frac{e^{-f}}{\mathbb{E}_\nu [e^{-f}]} d\nu$$

According to theorem 3, there exists $(u_n) \in \mathcal{D}^i$ such that for every $n \in \mathbb{N}$

$$\begin{aligned} \frac{dW^{u_n} \nu}{d\nu} \log \frac{dW^{u_n} \nu}{d\nu} &\rightarrow \frac{d\theta_0}{d\nu} \log \frac{d\theta_0}{d\nu} \\ \frac{dW^{u_n} \nu}{d\nu} \log \frac{d\theta_0}{d\nu} &\rightarrow \frac{d\theta_0}{d\nu} \log \frac{d\theta_0}{d\nu} \end{aligned}$$

in $L^1(\nu)$.

Since W^{u_n} is ν -a.s. invertible, we have

$$\mathbb{E}_\nu \left[f \circ W^{u_n} + \frac{1}{2} |u_n|_H^2 \right] = \mathbb{E}_\nu \left[f \frac{dW^{u_n} \nu}{d\nu} \right] + \mathbb{E}_\nu \left[\frac{dW^{u_n} \nu}{d\nu} \log \frac{dW^{u_n} \nu}{d\nu} \right]$$

When n goes to infinity, we have

$$\mathbb{E}_\nu \left[\frac{dW^{u_n} \nu}{d\nu} \log \frac{dW^{u_n} \nu}{d\nu} \right] \rightarrow \mathbb{E}_\nu \left[\frac{d\theta_0}{d\nu} \log \frac{d\theta_0}{d\nu} \right]$$

and since $f = -\log \frac{d\theta_0}{d\nu} - \log \mathbb{E}_\nu [e^{-f}]$,

$$\mathbb{E}_\nu \left[f \frac{dW^{u_n} \nu}{d\nu} \right] \rightarrow \mathbb{E}_\nu \left[f \frac{d\theta_0}{d\nu} \right]$$

So finally, when n goes to infinity,

$$\begin{aligned} \mathbb{E}_\nu \left[f \circ W^{u_n} + \frac{1}{2} |u_n|_H^2 \right] &\rightarrow \mathbb{E}_{\theta_0} [f] + H(\theta_0 | \nu) \\ &= -\log \mathbb{E}_\nu [e^{-f}] \end{aligned}$$

which conclude the proof. □

4.3. Retrieving the Prékopa-Leindler theorem.

Definition 3. We denote

$$H_b = \left\{ h \in H, \dot{h} \text{ is } dt - a.s. \text{ bounded} \right\}$$

Remark: Observe that $H_b \subset \mathcal{D}$ and that if $u \in \mathcal{D}$, $u(w) \in H_b$ ν -a.s.

Theorem 6. Assume that for any $u \in \mathcal{D}$,

$$W^u(w) = W^{u(w)}(w) \quad \nu - a.s.$$

Set $a, b, c : \mathbb{W} \rightarrow \mathbb{R}$ positive and measurable such that for every $h, k \in H_b$ and $t \in [0, 1]$ we have ν -a.s.

$$a \circ W^{th+(1-t)k} \exp\left(-\frac{1}{2}|th+(1-t)k|_H^2\right) \geq \left(b \circ W^h \exp\left(-\frac{1}{2}|h|_H^2\right)\right)^t \left(c \circ W^k \exp\left(-\frac{1}{2}|k|_H^2\right)\right)^{1-t}$$

then for any density d such that $h \in H_b \mapsto -\log d \circ W^h$ is ν -a.s. concave, if θ denotes the measure on \mathbb{W} given by $\frac{d\theta}{d\nu} = d$, we have in \mathbb{R} :

$$\mathbb{E}_\theta[a] \geq (\mathbb{E}_\theta[b])^t (\mathbb{E}_\theta[c])^{1-t}$$

Proof: First observe that eventually replacing a, b, c with da, db, dc we only need to prove the case $d = 1$ i.e. $\theta = \nu$

With the convention $\log(\infty) = \infty$ and $\log(0) = -\infty$, we denote

$$\tilde{a} = -\log a, \tilde{b} = -\log b, \tilde{c} = -\log c$$

We begin with the case where there exists $m, M > 0$ such that we have ν -a.s.

$$m \leq \tilde{a}, \tilde{b}, \tilde{c} \leq M$$

Set $t \in [0, 1]$, for $h, k \in H_b$, we have ν -a.s.

$$\begin{aligned} & a \circ W^{th+(1-t)k} \exp\left(-\frac{1}{2}|th+(1-t)k|_H^2\right) \\ & \geq \left(b \circ W^h \exp\left(-\frac{1}{2}|h|_H^2\right)\right)^t \left(c \circ W^k \exp\left(-\frac{1}{2}|k|_H^2\right)\right)^{1-t} \end{aligned}$$

So for $u_1, u_2 \in \mathcal{D}^i$

$$\begin{aligned} & a \circ W^{tu_1+(1-t)u_2} \exp\left(-\frac{1}{2}|tu_1+(1-t)u_2|_H^2\right) \\ & \geq \left(b \circ W^{u_1} \exp\left(-\frac{1}{2}|u_1|_H^2\right)\right)^t \left(c \circ W^{u_2} \exp\left(-\frac{1}{2}|u_2|_H^2\right)\right)^{1-t} \end{aligned}$$

hence applying the logarithm function, changing the sign and taking the expectation

$$\begin{aligned} & \mathbb{E}_\nu \left[\tilde{a} \circ W^{tu_1+(1-t)u_2} + \frac{1}{2}|tu_1+(1-t)u_2|_H^2 \right] \\ & \leq t \mathbb{E}_\nu \left[\tilde{b} \circ W^{u_1} + \frac{1}{2}|u_1|_H^2 \right] + (1-t) \mathbb{E}_\nu \left[\tilde{c} \circ W^{u_2} + \frac{1}{2}|u_2|_H^2 \right] \end{aligned}$$

So

$$\inf_{u \in \mathcal{D}^i} \mathbb{E}_\nu \left[\tilde{a} \circ W^u + \frac{1}{2} |u|_H^2 \right] \leq t \mathbb{E}_\nu \left[\tilde{b} \circ W^{u_1} + \frac{1}{2} |u_1|_H^2 \right] + (1-t) \mathbb{E}_\nu \left[\tilde{c} \circ W^{u_2} + \frac{1}{2} |u_2|_H^2 \right]$$

According to theorem 5 we have

$$-\log \mathbb{E}_\nu [e^{-\tilde{a}}] \leq t \mathbb{E}_\nu \left[\tilde{b} \circ W^{u_1} + \frac{1}{2} |u_1|_H^2 \right] + (1-t) \mathbb{E}_\nu \left[\tilde{c} \circ W^{u_2} + \frac{1}{2} |u_2|_H^2 \right]$$

which implies

$$\begin{aligned} -\log \mathbb{E}_\nu [e^{-\tilde{a}}] &\leq t \mathbb{E}_\nu \left[\tilde{b} \circ W^{u_1} + \frac{1}{2} |u_1|_H^2 \right] + (1-t) \inf_{v \in \mathcal{D}^i} \mathbb{E}_\nu \left[\tilde{c} \circ W^v + \frac{1}{2} |v|_H^2 \right] \\ &= t \mathbb{E}_\nu \left[\tilde{b} \circ W^{u_1} + \frac{1}{2} |u_1|_H^2 \right] - (1-t) \log \mathbb{E}_\nu [e^{-\tilde{c}}] \end{aligned}$$

which implies once again

$$\begin{aligned} -\log \mathbb{E}_\nu [e^{-\tilde{a}}] &\leq \inf_{v \in \mathcal{D}^i} \left(t \mathbb{E}_\nu \left[\tilde{b} \circ W^v + \frac{1}{2} |v|_H^2 \right] \right) - (1-t) \log \mathbb{E}_\nu [e^{-\tilde{c}}] \\ &= -t \log \mathbb{E}_\nu [e^{-\tilde{b}}] - (1-t) \log \mathbb{E}_\nu [e^{-\tilde{c}}] \end{aligned}$$

taking the opposite and applying the exponential, we get

$$\mathbb{E}_\nu [e^{-\tilde{a}}] \geq \left(\mathbb{E}_\nu [e^{-\tilde{b}}] \right)^t \left(\mathbb{E}_\nu [e^{-\tilde{c}}] \right)^{1-t}$$

For the general case, denote for $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$

$$\begin{aligned} \tilde{a}_n &= \tilde{a} \wedge n, \tilde{b}_n = \tilde{b} \wedge n, \tilde{c}_n = \tilde{c} \wedge n \\ \tilde{a}_{nm} &= \tilde{a}_n + \frac{1}{m}, \tilde{b}_{nm} = \tilde{b}_n + \frac{1}{m}, \tilde{c}_{nm} = \tilde{c}_n + \frac{1}{m} \end{aligned}$$

For every $h, k \in H_b$, we have ν -a.s.:

$$\tilde{a}_{nm} \circ W^{th+(1-t)k} + \frac{1}{2} |th+(1-t)k|_H^2 \leq t \tilde{b}_{nm} \circ W^h + \frac{1}{2} |h|_H^2 + (1-t) \tilde{c}_{nm} \circ W^k + \frac{1}{2} |k|_H^2$$

so the bounded case we treated above ensures that

$$\mathbb{E}_\nu [e^{-\tilde{a}_{nm}}] \geq \left(\mathbb{E}_\nu [e^{-\tilde{b}_{nm}}] \right)^t \left(\mathbb{E}_\nu [e^{-\tilde{c}_{nm}}] \right)^{1-t}$$

The monotone limit theorem enables us to take the limit with relation to m and then to take it again with respect to n to get the result. \square

5. Extension of the map $u \mapsto W^u$

5.1. Extension of the map $u \mapsto W^u$ for invertibility results. Invertibility results can give stochastic differential equations solutions in certain cases, so it can be useful to extend these results to a larger domain.

Definition 4. Set $\tilde{\mathcal{D}}$ a subset of $G_0(\nu, \beta)$ such that the map $u \mapsto W^u$ can be extended to $\tilde{\mathcal{D}}$ while verifying the following conditions:

- (i) $\mathcal{D} \subset \tilde{\mathcal{D}} \subset G_0(\nu, \beta)$
- (ii) For every $u \in \tilde{\mathcal{D}}$, W^u is adapted.
- (iii) For every $u \in \tilde{\mathcal{D}}$, the law of W^u under $\tilde{\nu}^u$ is the same as the law of W under ν , where $\tilde{\nu}^u$ is

defined by $\frac{d\tilde{\nu}}{d\nu} = \rho(-\delta_\beta u)$

(iv) For every $u \in \tilde{\mathcal{D}}$,

$$\beta \circ W^u = \beta + u$$

(v) For every $u, v \in \tilde{\mathcal{D}}$ such that $v + u \circ W^v \in \tilde{\mathcal{D}}$

$$W^u \circ W^v = W^{v+u \circ W^v} \nu - a.s.$$

(vi) There exists $\tilde{\tilde{\mathcal{D}}}$ such that $\tilde{\mathcal{D}} \subset \tilde{\tilde{\mathcal{D}}} \subset L_a^0(\nu, H)$, $\tilde{\mathcal{D}} = \tilde{\tilde{\mathcal{D}}} \cap G_0(\nu, \beta)$ and for every $u \in \tilde{\mathcal{D}}$ such that the equation $u + v \circ W^u$ has a solution in $G_0(\nu, \beta)$, this equation has a solution in $\tilde{\tilde{\mathcal{D}}}$.

Proposition 6. Set $u \in \tilde{\mathcal{D}}$, for every bounded measurable $f : W \rightarrow \mathbb{R}$

$$\mathbb{E}_\nu[f] = \mathbb{E}_\nu[f \circ W^u \rho(-\delta_\beta u)]$$

Moreover

$$W^u \nu \sim \nu$$

Proof: The proof is the same as the case $u \in \mathcal{D}$, see section 2. □

Remark: \mathcal{D} verify the set of conditions above.

Theorem 7. For every $u \in \tilde{\mathcal{D}} \cap L^2(\nu, H)$, we have

$$H(W^u \nu | \nu) \leq \frac{1}{2} \mathbb{E}_\nu[|u|_H^2]$$

and the three following propositions are equivalent:

$$1) H(W^u \nu | \nu) = \frac{1}{2} \mathbb{E}_\nu[|u|_H^2]$$

2) there exists $v \in \tilde{\mathcal{D}}$ such that

$$W^u \circ W^v = W^v \circ W^u = I_{\mathbb{W}} \nu - a.s.$$

$$\frac{dW^u \nu}{d\nu} = \rho(-\delta_\beta v)$$

$$\frac{dW^v \nu}{d\nu} = \rho(-\delta_\beta u)$$

3) W^u is ν -a.s. left-invertible

Proof: Set $u \in \mathcal{D}$, we prove that $H(W^u \nu | \nu) \leq \frac{1}{2} \mathbb{E}_\nu[|u|_H^2]$ exactly as in proposition 4

For the second assertion, we prove (1) \Rightarrow (2) first. Exactly as in the proof of theorem 2, we obtain

$$(5.5) \quad L \circ W^u \rho(-\delta_\beta u) = 1$$

Since β is a ν -Brownian motion, there exists $v \in G_0(\nu, \beta)$ such that $L = \rho(-\delta_\beta v)$.

We apply the exponential to 5.5 to get:

$$0 = \delta_\beta v \circ W^u + \frac{1}{2} |v \circ W^u|_H^2 + \delta_\beta u + |u|_H^2$$

We have:

$$\delta_\beta v \circ W^u = \int_0^1 \dot{v}(s) \circ W^u d\beta(s) + \langle v \circ W^u, u \rangle_H$$

so finally we have:

$$(5.6) \quad 0 = \delta_\beta(v \circ W^u + u) + \frac{1}{2}|v \circ W^u + u|_H^2$$

According to Girsanov theorem $\beta + v$ is a $W^u\nu$ -Brownian motion, so:

$$\begin{aligned} \mathbb{E}_\nu [L \log L] &= \mathbb{E}_{W^u\nu} [\log L] \\ &= \mathbb{E}_{W^u\nu} \left[-\int_0^1 \dot{v}(s) d\beta(s) - \frac{1}{2} \int_0^1 |\dot{v}(s)|^2 ds \right] \\ &= \frac{1}{2} \mathbb{E}_{W^u\nu} \left[\int_0^1 |\dot{v}(s)|^2 ds \right] \\ &= \frac{1}{2} \mathbb{E}_\nu [|v \circ W^u|_H^2] \end{aligned}$$

So $v \circ W^u \in L_a^2(\nu, H)$ and we can take the expectation with respect to ν in 5.6 to obtain $u + v \circ W^u = 0$ ν -a.s. Condition (vi) gives the existence of $\tilde{v} \in \tilde{\mathcal{D}}$ such that ν -a.s.

$$u + \tilde{v} \circ W^u = 0$$

We have $v = \tilde{v} \circ W^u$ ν -a.s. so $v = \tilde{v}$ ν -a.s. since $W^u\nu \sim \nu$, which implies $v \in \tilde{\mathcal{D}}$ and condition (iv) gives ν -a.s.

$$W^v \circ W^u = I_{\mathbb{W}}$$

Proposition 3 gives ν -a.s.

$$W^u \circ W^v = I_{\mathbb{W}}$$

Finally, set $f \in C_b(\mathbb{W})$,

$$\begin{aligned} \mathbb{E}_\nu [f \circ W^u] &= \mathbb{E}_\nu [f \circ W^u \circ W^v \rho(-\delta_\beta v)] \\ &= \mathbb{E}_\nu [f \rho(-\delta_\beta v)] \\ \mathbb{E}_\nu [f \circ W^v] &= \mathbb{E}_\nu [f \circ W^v \circ W^u \rho(-\delta_\beta u)] \\ &= \mathbb{E}_\nu [f \rho(-\delta_\beta u)] \end{aligned}$$

which gives

$$\begin{aligned} \frac{dW^u\nu}{d\nu} &= \rho(-\delta_\beta v) \\ \frac{dW^v\nu}{d\nu} &= \rho(-\delta_\beta u) \end{aligned}$$

(2) \Rightarrow (3) is immediate. Now we prove (3) \Rightarrow (1). We still denote $L = \frac{dW^u\nu}{d\nu}$.

Assume that W^u admits a left inverse V . Set $v = -u \circ V$.

We have ν -a.s.

$$v \circ W^u = -u$$

and

$$\mathbb{E}_{W^u\nu} \left[1_{\int_0^1 |\dot{v}(s)|^2 ds < \infty} \right] = \mathbb{E}_\nu \left[1_{\int_0^1 |\dot{u}(s)|^2 ds < \infty} \right] = 1$$

so $v \in L^0(W^u\nu, H)$ and $v \in L^0(\nu, H)$ since $W^u\nu \sim \nu$.

Now set $\dot{v}^n = \max(n, \min(\dot{v}, -n))$, $\dot{v}^n \circ W^u$ is adapted. Set $A \in L^2(dt \times d\nu)$ an adapted process, we have:

$$\begin{aligned} \mathbb{E}_\nu \left[\rho(-\delta_\beta u) \int_0^1 \dot{v}^n(s) \circ W^u A(s) \circ W^u ds \right] &= \mathbb{E}_\nu \left[\int_0^1 \dot{v}^n(s) A(s) ds \right] \\ &= \mathbb{E}_\nu \left[\int_0^1 \mathbb{E}_\nu [\dot{v}^n(s) | \mathcal{F}(s)] A(s) ds \right] \\ &= \mathbb{E}_\nu \left[\rho(-\delta_\beta u) \int_0^1 \mathbb{E}_\nu [\dot{v}^n(s) | \mathcal{F}(s)] \circ W^u A(s) \circ W^u ds \right] \end{aligned}$$

So $\mathbb{E}_\nu [\dot{v}^n(s) | \mathcal{F}(s)] \circ W^u = \dot{v}^n(s) \circ W^u ds \times d\nu$ -a.s. which implies $\mathbb{E}_\nu [\dot{v}^n(s) | \mathcal{F}(s)] = \dot{v}^n(s) ds \times d\nu$ -a.s. since $W^u\nu \sim \nu$.

An algebraic calculation gives ν -a.s.

$$\rho(-\delta_\beta v) \circ W^u \rho(-\delta_\beta u) = 1$$

Now set $g \in C_b(W, \mathbb{R}_+)$, we have:

$$\begin{aligned} \mathbb{E}_\nu [gL] &= \mathbb{E}_\nu [g \circ W^u] \\ &= \mathbb{E}_\nu [g \circ W^u \rho(-\delta_\beta v) \circ W^u \rho(-\delta_\beta u)] \\ &\leq \mathbb{E}_\nu [g \rho(-\delta_\beta v)] \end{aligned}$$

So $L \leq \rho(-\delta_\beta v)$ ν -a.s. and since $\mathbb{E}_\nu [\rho(-\delta_\beta v)] = 1$, we have

$$L \circ W^u \rho(-\delta_\beta u) = 1$$

and we can compute $H(W^u\nu|\nu)$:

$$\begin{aligned} H(W^u\nu|\nu) &= \mathbb{E}_\nu [L \log L] \\ &= \mathbb{E}_\nu [\log L \circ W^u] \\ &= \mathbb{E}_\nu [-\log \rho(-\delta_\beta u)] \\ &= \frac{1}{2} \mathbb{E}_\nu [|u_H^2|] \end{aligned}$$

□

As in section 3.2, the proof of theorem 7 give the following additional results.

Corollary 4. *Set $u \in \tilde{\mathcal{D}}$, we have*

$$L \circ W^u \mathbb{E}_\nu \left[\rho(-\delta_\beta u) | \mathcal{F}_1^{W^u} \right] = 1$$

and if W^u is ν -a.s. left-invertible, we have

$$L \circ W^u \rho(-\delta_\beta u) = 1$$

In certain cases invertibility results lead to the existence of a strong solutions of stochastic differential equations.

Theorem 8. Assume that for every $u \in \tilde{\mathcal{D}}$, we can write ν -a.s.

$$W^u = I_{\mathbb{W}} + w^u$$

with $w^u \in L_a^0(\nu, H)$.

Set $u \in \tilde{\mathcal{D}}$,

$$H(W^u \nu | \nu) = \frac{1}{2} \mathbb{E}_{\nu} [|u|_H^2]$$

if and only if there exists $v \in \tilde{\mathcal{D}}$ such that W^v is a strong solution to

$$dW^v(t) = - \overbrace{w^u}^{\cdot}(t) \circ W^v dt + dW(t)$$

Proof: Assume that $H(W^u \nu | \nu) = \frac{1}{2} \mathbb{E}_{\nu} [|u|_H^2]$, according to theorem 7, there exists $v \in \tilde{\mathcal{D}}$ such that ν -a.s.

$$W^v \circ W^u = W^u \circ W^v = W$$

Since $W^u = W + w^u$, we have

$$W^v + w^u \circ W^v = W$$

and W^v is a strong solution to

$$dW^v(t) = - \overbrace{w^u}^{\cdot}(t) \circ W^v dt + dW(t)$$

Conversely, assume the existence of $v \in \tilde{\mathcal{D}}$ such W^v is a strong solution to

$$dW^v(t) = - \overbrace{w^u}^{\cdot}(t) \circ W^v dt + dW(t)$$

We have $W^u \circ W^v = I_W$ ν -a.s., and $W^v \circ W^u = I_W$ $W^v \nu$ -a.s. Since $W^v \nu \sim \nu$, we can conclude with theorem 7. \square

5.2. Extension of the map $u \mapsto W^u$ for variational calculus.

Theorem 9. For every measurable function $f : \mathbb{W} \rightarrow \mathbb{R}$ such that $\mathbb{E}_{\nu} [(|f| + 1)e^{-f}] < \infty$ and

$$-\log \mathbb{E}_{\nu} [e^{-f}] = \inf_{u \in \mathcal{D}^i} \mathbb{E}_{\nu} \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right]$$

we have

$$-\log \mathbb{E}_{\nu} [e^{-f}] = \inf_{u \in \tilde{\mathcal{D}} \cap L_a^2(\nu, H)} \mathbb{E}_{\nu} \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right]$$

Proof: For $u \in \tilde{\mathcal{D}} \cap L_a^2(\nu, H)$, we have

$$\begin{aligned} -\log \mathbb{E}_{\nu} [e^{-f}] &\leq \mathbb{E}_{W^u \nu} [f] + H(W^u \nu | \nu) \\ &\leq \mathbb{E}_{\nu} \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right] \end{aligned}$$

So

$$\begin{aligned} -\log \mathbb{E}_{\nu} [e^{-f}] &\leq \inf_{u \in \tilde{\mathcal{D}} \cap L_a^2(\nu, H)} \mathbb{E}_{\nu} \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right] \\ &\leq \inf_{u \in \mathcal{D}^i} \mathbb{E}_{\nu} \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right] \end{aligned}$$

□

Theorem 10. *Set $f : \mathbb{W} \rightarrow \mathbb{R}$ a measurable function verifying $\mathbb{E}_\nu [|f|(1 + e^{-f})] < \infty$, then if there exists some $u \in \tilde{\mathcal{D}} \cap L_a^2(\nu, H)$ such that W^u is ν -a.s. left-invertible and $\frac{dW^u\nu}{d\nu} = \frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}]}$, then we have*

$$-\log \mathbb{E}_\nu [e^{-f}] = \inf_{u \in \tilde{\mathcal{D}} \cap L_a^2(\nu, H)} \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right]$$

Proof: Since W^u is ν -a.s. left invertible and that $\frac{dW^u\nu}{d\nu} = \frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}]}$. We have

$$\frac{1}{2} \mathbb{E}_\nu [|u|_H^2] = H(W^u\nu|\nu) = \mathbb{E}_\nu \left[\frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}]} \log \left(\frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}]} \right) \right]$$

and

$$\begin{aligned} \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right] &= \mathbb{E}_\nu \left[\frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}]} f + \frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}]} \log \left(\frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}]} \right) \right] \\ &= -\log \mathbb{E}_\nu [e^{-f}] \end{aligned}$$

and we conclude the proof with last proposition. □

Theorem 11. *Set $f : \mathbb{W} \rightarrow \mathbb{R}$ a measurable function such that*

$$-\log \mathbb{E}_\nu [e^{-f}] = \inf_{u \in \tilde{\mathcal{D}} \cap L_a^2(\nu, H)} \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right]$$

Denote this infimum J_ . It is attained at $u \in \tilde{\mathcal{D}} \cap L_a^2(\nu, H)$ if and only if W^u is ν -a.s. left-invertible and $\frac{dW^u\nu}{d\nu} = \frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}]}$.*

Proof: The direct implication is given by last theorem. Conversely, if W^u is not ν -a.s. left-invertible, $H(W^u\nu|\nu) < \frac{1}{2} \mathbb{E}_\nu [|u|_H^2]$ and

$$\begin{aligned} -\log \mathbb{E}_\nu [e^{-f}] &= \inf_{\theta \in \mathcal{P}} (\mathbb{E}_\theta[f] + H(\theta|\nu)) \leq \inf_{\alpha \in \tilde{\mathcal{D}} \cap L_a^2(\nu, H)} \mathbb{E}_{W^\alpha\nu} [f] + H(W^\alpha\nu|\nu) \\ &\leq \mathbb{E}_{W^u\nu} [f] + H(W^u\nu|\nu) \\ &< \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \right] \end{aligned}$$

which is a contradiction.

We get $\frac{dW^u\nu}{d\nu} = L$ by uniqueness of the minimizing measure of $\inf_{\theta \in \mathcal{P}} (\mathbb{E}_\theta[f] + H(\theta|\nu))$. □

6. Examples

In this section we discuss several examples that fit into the framework we elaborated. Each time, we prove that the conditions of section 2 and definition 4 are satisfied. This ensures that every result from section 2 to 5 apply, except theorems 6 and 8 which require additional hypothesis. We also discuss whether these theorems apply or not.

6.1. Diffusion. Set $m \leq d \in \mathbb{N}^*$ such that $m + d = n$, $c \in \mathbb{R}$, $\sigma : \mathbb{R}^m \rightarrow \mathcal{M}_{m,d}(\mathbb{R})$ and $b : \mathbb{R}^m \rightarrow \mathbb{R}^m$ bounded and lipschitz functions. σ_i will denote the i -th column of σ . Notice that every matrix will be identified with its canonical linear operator. Set $(\Omega, \theta, (\mathcal{G}_t))$ a probability space, V a θ -Brownian motion on Ω with values in \mathbb{R}^d . Set Y a \mathbb{R}^m -valued strong solution of the stochastic differential equation:

$$Y(t) = c + \int_0^t \sigma(Y(s))dV(s) + \int_0^t b(Y(s))ds$$

on $(\Omega, \theta, (\mathcal{G}_t), B)$. The hypothesis on σ and b ensure the existence and uniqueness of Y if we impose its paths to be continuous.

We denote μ the Wiener measure on $C([0, 1], \mathbb{R}^d)$ and μ^X the image measure of Y .

We define the processes X and B on \mathbb{W} by:

$$\begin{aligned} X(t) &: (w, w') \in \mathbb{W} \mapsto w(t) \in \mathbb{R}^m \\ B(t) &: (w, w') \in \mathbb{W} \mapsto w'(t) \in \mathbb{R}^d \end{aligned}$$

Proposition 7. *Under $\mu^X \times \mu$, the law of X is μ^X , B is a Brownian motion and they are independent. There exists θ, η such that if we define $\beta_{\mathbb{X}}$ as*

$$\beta_{\mathbb{X}} = \int_0^\cdot \theta(X(s))dM(s) + \int_0^\cdot \eta(X(s))dB(s)$$

$\beta_{\mathbb{X}}$ is a $\mu^X \times \mu$ -Brownian motion and $\mu^X \times \mu$ -a.s.

$$X = c + \int_0^\cdot \sigma(X(s))d\beta_{\mathbb{X}}(s) + \int_0^\cdot b(X(s))ds$$

Proof: See [9]. □

This construction of $\beta_{\mathbb{X}}$ is taken from [13].

Definition 5. *We denote*

$$\mathbb{X} = (X, \beta_{\mathbb{X}})$$

and $\mu^{\mathbb{X}}$ its image measure.

X is a $\mu^{\mathbb{X}}$ path-continuous strong solution of the stochastic differential equation

$$X = c + \int_0^\cdot \sigma(X(s))d\beta_{\mathbb{X}}(s) + \int_0^\cdot b(X(s))ds$$

For $u \in G_0(\mu^{\mathbb{X}}, \beta_X)$, set $\beta_{\mathbb{X}}^u = \beta + u$ and X^u the $\mu^{\mathbb{X}}$ -a.s. path-continuous strong solution of the stochastic differential equation

$$X^u = c + \int_0^\cdot \sigma(X^u(s))d\beta_{\mathbb{X}}^u(s) + \int_0^\cdot b(X^u(s))ds$$

Finally, we denote

$$\mathbb{X}^u = (X^u, \beta_{\mathbb{X}} + u)$$

Theorem 12. $(\mathbb{W}, \mu^{\mathbb{X}}, \beta_{\mathbb{X}}, (\mathbb{X}^u)_{u \in \mathcal{D}})$ verify the conditions of section 2. $(\mathbb{W}, \mu_a, \beta_{\mathbb{X}}, (\mathbb{X}^u)_{u \in G_0(\mu^{\mathbb{X}}, \beta_{\mathbb{X}})})$ verify the conditions of definition 4.

Proof: See [9]. □

Corollary 5. *It is clear that for every $u \in \mathcal{D}$, we clearly have $\mu^{\mathbb{X}}$ -a.s.*

$$\mathbb{X}^u(w) = \mathbb{X}^{u(w)}(w)$$

so theorem 6 applies.

Corollary 6. *Assume that $\sigma \in \mathbb{R}$, then theorem 8 applies. Set $u \in G_2(\mu^{\mathbb{X}}, \beta_{\mathbb{X}})$ and denote*

$$\dot{w}_{\mathbb{X}}^u(t) = (\dot{u}(t) + b(X^u(t)) - b(X(t)), \dot{u}(t))$$

We have $\mu^{\mathbb{X}}$ -a.s.

$$\mathbb{X}^u = I_{\mathbb{W}} + w_{\mathbb{X}}^u$$

so

$$H(\mathbb{X}^u \mu^{\mathbb{X}} | \mu^{\mathbb{X}}) = \frac{1}{2} \mathbb{E}_{\mu^{\mathbb{X}}} [|u|_H^2]$$

if and only if there exists $v \in G_0(\mu^{\mathbb{X}}, \beta_{\mathbb{X}})$ such that \mathbb{X}^v is a strong solution to the stochastic differential equation:

$$d\mathbb{X}^v(t) = -\dot{w}_{\mathbb{X}}^u(t)dt \circ \mathbb{X}^v + dW(t)$$

6.2. Brownian bridge. We still denote μ the Wiener measure on \mathbb{W} . Set $a \in \mathbb{R}^n$, we denote μ_a the measure on \mathbb{W} such that for any bounded measurable function f we have

$$\mathbb{E}_{\mu_a} [f] = \mathbb{E}_{\mu} [f | W_1 = a]$$

μ_a can also be defined as follow : let \mathcal{E}_a be the Dirac measure in a , $\mathcal{E}_a(W_1)$ is a positive Wiener distributions hence it defines a Radon measure ν_a on \mathbb{W} , then

$$\mu_a = \left(\frac{1}{2\pi} \right)^n \nu_a$$

We recall the definition of a Brownian bridge:

Definition 6. *Set (Ω, \mathcal{G}, Q) a probability space. An a -Brownian bridge X under a probability Q is a path-continuous Gaussian process such that $\mathbb{E}_Q [X(t)] = at$ and $\text{cov}(X(s), X(t)) = ((s \wedge t) - st) I_d$*

Proposition 8. *W is an a -Brownian bridge under μ_a , and the process β_a defined as*

$$\beta_a(t) = W(t) - at + \int_0^t \frac{W(s) - as}{1-s} ds$$

is a Brownian motion under μ_a and the filtrations of β_a and W completed with respect to μ_a are equal. Moreover, we have

$$W(t) = at + (1-t) \int_0^t \frac{d\beta_a(s)}{1-s}$$

Proof: It is easy to verify that the process Z given by $Z(t) = W(t) + at - tW_1$ is an a -Brownian bridge under μ and is independent of W_1 . If f is a bounded continuous function on \mathbb{W} , we have

$$\begin{aligned} \mathbb{E}_{\mu} [f(Z)] &= \mathbb{E}_{\mu} [f(Z) | W_1 = a] \\ &= \mathbb{E}_{\mu} [f(W) | W_1 = a] \\ &= \mathbb{E}_{\mu} [f | W_1 = a] \\ &= \mathbb{E}_{\mu_a} [f] \\ &= \mathbb{E}_{\mu_a} [f(W)] \end{aligned}$$

So W is indeed an a -Brownian bridge under μ_a .

Now consider the process X given by

$$X(t) = (1-t)W\left(\frac{t}{1-t}\right) + at$$

It is easy again to verify that X is a Brownian bridge under μ . Now consider

$$M = \int_0^\cdot \frac{dW(s)}{1-s}$$

M is a continuous martingale under μ and

$$\langle M_i, M_j \rangle(t) = \delta_{ij} \int_0^t \frac{ds}{(1-s)^2} = \delta_{ij} \frac{t}{1-t}$$

So the Dubins-Schwartz theorem ensures that M and $W(\frac{\cdot}{1-\cdot})$ have the same distribution under μ , so X and \tilde{X} have the same distribution, where \tilde{X} is given by

$$\begin{aligned} \tilde{X}(t) &= (1-t) \int_0^t \frac{dW(s)}{1-s} + at \\ &= W(t) - \int_0^t \frac{\tilde{X}(s) - as}{1-s} ds + at \end{aligned}$$

the last equality coming from Ito's formula.

W is a Brownian motion under μ and the law of (\tilde{X}, W) under μ is the same as the law of (W, β_a) under μ_a so $(\beta_a(t), t \in [0, 1])$ is a Brownian motion under μ_a .

We recall that the filtrations we are considering are all completed with respect to μ_a . From the expression of β_a , we have clearly for any $t \in [0, 1]$,

$$\mathcal{F}_t^{\beta_a} \subset \mathcal{F}_t^W$$

Furthermore, $s \mapsto \frac{1}{1-s}$ being lipschitz on any $[0, t]$ with $t < 1$, W is the strong solution of a stochastic differential equation relative to β_a and we have for any $t < 1$,

$$\mathcal{F}_t^W \subset \mathcal{F}_t^{\beta_a}$$

W being μ_a -a.s. path continuous, we have

$$\bigcup_{t < 1} \mathcal{F}_t^W = \mathcal{F}_1^W$$

On the other hand, since $(\beta_a(t), t \in [0, 1])$ is a Brownian motion, $(\langle \beta_a \rangle(t), t \in [0, 1])$ is bounded by 1, so $\beta_a(t)$ converges μ_a -a.s. and in $L^2(\mu_a, \mathbb{R}^n)$. Denote $\beta_a(1)$ the limit, for $s \in [0, 1]$ we have

$$Cov(\beta_a(s), \beta_a(t)) = \lim_{t' \rightarrow 1} Cov(\beta_a(s), \beta_a(t')) = s \wedge t'$$

So

$$\bigcup_{t < 1} \mathcal{F}_t^{\beta_a} = \mathcal{F}_1^{\beta_a}$$

$(\beta_a(t), t \in [0, 1])$ is a μ_a -Brownian motion and β_a and W have the same filtration. \square

The following remark will be useful in next section.

Remark: For $a \in \mathbb{R}^n$ and $t \in [0, 1]$, we have μ_a -a.s.

$$\beta_a(t) = W_t + \int_0^t \frac{W_s - a}{1-s} ds$$

Definition 7. For $u \in G_0(\mu_a, \beta_a)$, we denote $\beta_a^u = \beta_a + u$.

Proposition 9. Set $u \in G_0(\mu_a, \beta_a)$, then there exists a unique μ_a -a.s. path continuous process W_a^u such that

$$W_a^u(t) = \beta_a^u(t) + at - \int_0^t \frac{W_a^u(s) - as}{1-s} ds$$

Furthermore, we have

$$\begin{aligned} W_a^u(t) &= at + (1-t) \int_0^t \frac{d\beta_a^u(s)}{1-s} \\ &= W(t) + \int_0^t \left(\dot{u}(s) - \int_0^s \frac{\dot{u}(r)}{1-r} dr \right) ds \end{aligned}$$

Proof: Set $u \in G_0(\mu_a, \beta_a)$, for $t < 1$, straight calculation gives

$$at + (1-t) \int_0^t \frac{d\beta_a^u(s)}{1-s} = W + \int_0^t \left(\dot{u}(s) - \int_0^s \frac{\dot{u}(r)}{1-r} dr \right) ds$$

Define W_a^u on $[0, 1)$ as

$$W_a^u(t) = at + (1-t) \int_0^t \frac{d\beta_a^u(s)}{1-s}$$

The Ito formula gives

$$W_a^u(t) = \beta_a^u(t) + at - \int_0^t \frac{W_a^u(s) - as}{1-s} ds$$

$x \mapsto \frac{1}{1-x}$ being lipschitz on every $[0, t]$ for $t < 1$, the μ_a -a.s. pathwise uniqueness is true on every $[0, t]$, hence on $[0, 1)$.

It remains to prove that there is no explosion in 1. Set $\tilde{\mu}_a^u$ the measure on \mathbb{W} defined by

$$\frac{d\tilde{\mu}_a^u}{d\mu_a} = \rho(-\delta_{\beta_a} u)$$

Since $u \in G_0(\mu_a, \beta_a)$, it is clear that the law of W_a^u under $\tilde{\mu}_a^u$ is the same as the law of W_a under μ_a so

$$\tilde{\mu}_a^u \left(\limsup_{t \rightarrow 1} |W_a^u(t)| = \infty \right) = \mu_a \left(\limsup_{t \rightarrow 1} |W_a(t)| = \infty \right) = 0$$

$\tilde{\mu}_a^u \sim \mu_a$ so

$$\mu_a \left(\limsup_{t \rightarrow 1} |W_a^u(t)| = \infty \right) = 0$$

□

Theorem 13. $(\mathbb{W}, \mu_a, \beta_a, (W_a^u)_{u \in \mathcal{D}})$ verify the conditions of section 2. $(\mathbb{W}, \mu_a, \beta_a, (W_a^u)_{u \in G_0(\mu_a, \beta_a)})$ verify the conditions of definition 4.

Proof: We have $W_a^0 = W$ and β_a is a μ_a -Brownian motion. Now we just have to verify the conditions of definition 4 since those imply conditions (iii) to (v) of section 2.

(i), (ii) and (iii) are clear, so is (vi) taking $\widetilde{\mathcal{D}} = G_0$.

Set $u \in G_0(\mu_a, \beta_a)$, we have

$$\begin{aligned}\beta_a \circ W_a^u(t) &= \left(W(t) - at + \int_0^t \frac{W(s) - as}{1-s} ds \right) \circ W_a^u \\ &= W_a^u(t) - at + \int_0^t \frac{W_a^u(s) - as}{1-s} ds \\ &= \beta_a^u(t)\end{aligned}$$

so condition (iv) is verified. Now set $v \in G_0(\mu_a, \beta_a)$ such that $v + u \circ W^v \in G_0(\mu_a, \beta_a)$, we have

$$\begin{aligned}W_a^u(t) \circ W_a^v &= \left(W(t) + \int_0^t \left(\dot{u}(s) - \int_0^s \frac{\dot{u}(r)}{1-r} dr \right) ds \right) \circ W_a^v \\ &= W_a^v(t) + \int_0^t \left(\dot{u}(s) \circ W_a^v - \int_0^s \frac{\dot{u}(r) \circ W_a^v}{1-r} dr \right) ds \\ &= W(t) + \int_0^t \left(\dot{v}(s) + \dot{u}(s) \circ W_a^v - \int_0^s \frac{\dot{v}(r) + \dot{u}(r) \circ W^v(r)}{1-r} dr \right) ds \\ &= W^{v+u \circ W_a^v}\end{aligned}$$

which gives condition (v). □

Corollary 7. *It is clear that for every $u \in \mathcal{D}$, we clearly have μ_a -a.s.*

$$W_a^u(w) = W_a^{u(w)}(w)$$

so theorem 6 applies.

Corollary 8. *Theorem 8 applies. Set $u \in G_2(\mu_a, \beta_a)$, we have μ_a -a.s.*

$$W_a^u = I_{\mathbb{W}} + \int_0^\cdot \dot{u}(t) - \int_0^t \frac{\dot{u}(s)}{1-s} ds dt$$

so

$$H(W_a^u \mu_a | \mu_a) = \frac{1}{2} \mathbb{E}_{\mu_a} [|u|_H^2]$$

if and only if there exists $v \in G_0(\mu_a, \beta_a)$ such that W_a^v is a strong solution to the stochastic differential equation:

$$dW_a^v(t) = - \left(\dot{u}(t) - \int_0^t \frac{\dot{u}(s)}{1-s} ds \right) dt \circ W_a^v + dW(t)$$

6.3. Loop measure. We keep the notations of last section. Denote

$$S = \{a \in \mathbb{R}^n, |a| = 1\}$$

and set $\alpha : S \rightarrow \mathbb{R}_+$ a locally lipschitz function such that $\{x, \alpha(x) \neq 0\}$ is of strictly positive measure for the Lebesgue measure on S and

$$\int_S \alpha(a) da = 1$$

We define the measure ν_l as follow: for any bounded measurable function f on \mathbb{W} , we set

$$\mathbb{E}_{\nu_l}[f] = \int_S \alpha(a) \mathbb{E}_{\mu_a}[f] da$$

For more on loop measures, see Fang's work in [6].

Definition 8. We denote

$$\begin{aligned} h_a : (t, x) &\in [0, 1) \times \mathbb{R}^n \mapsto \left(\frac{1}{\pi(1-t)} \right)^{\frac{n}{2}} \exp \left(\frac{-|x-a|^2}{2(1-t)} \right) \\ h : (t, x) &\in [0, 1) \times \mathbb{R}^n \mapsto \int_S \alpha(a) h_a(t, x) da \end{aligned}$$

Proposition 10. Set $a \in \mathbb{R}^n$ and $t \in [0, 1)$, then

$$\left. \frac{d\mu_a}{d\mu} \right|_{\mathcal{F}_t^W} = h_a(t, W_t)$$

Proof: For convenience we consider the case $n = 1$, the general proof is the same. Every \mathcal{F}_t^W measurable $f : \mathbb{W} \rightarrow \mathbb{R}$ is a function of $W(\cdot \wedge t)$, hence of $\beta_a(\cdot \wedge t)$. $\beta_a(\cdot \wedge t)$ is a Brownian motion on $[0, t]$ under μ_a . Now denote $\tilde{\mu}$ the probability measure on \mathbb{W} given by

$$\frac{d\tilde{\mu}}{d\mu} = \exp \left(- \int_0^t \frac{W(s) - a}{1-s} dW(s) - \frac{1}{2} \int_0^t \left(\frac{W(s) - a}{1-s} \right)^2 ds \right)$$

According to Girsanov theorem, $\beta_a(\cdot \wedge t)$ is also a Brownian motion under $\tilde{\mu}$ and

$$\begin{aligned} \left. \frac{d\mu_a}{d\mu} \right|_{\mathcal{F}_t^W} &= \left. \frac{d\tilde{\mu}}{d\mu} \right|_{\mathcal{F}_t^W} \\ &= \exp \left(- \int_0^t \frac{W(s) - a}{1-s} dW(s) - \frac{1}{2} \int_0^t \left(\frac{W(s) - a}{1-s} \right)^2 ds \right) \end{aligned}$$

Finally, Ito formula gives

$$\left(\frac{1}{\pi(1-t)} \right)^{\frac{1}{2}} \exp \left(\frac{-|W(s) - a|^2}{2(1-t)} \right) = \exp \left(- \int_0^t \frac{W(s) - a}{1-s} dW(s) - \frac{1}{2} \int_0^t \left(\frac{W(s) - a}{1-s} \right)^2 ds \right)$$

□

Proposition 11. Set $t \in [0, 1)$, we have

$$\left. \frac{d\nu}{d\mu} \right|_{\mathcal{F}_t^W} = h(t, W_t)$$

Proof: Set $C \in \mathcal{F}_t^W$, Fubini-Tonelli theorem gives

$$\begin{aligned} \mathbb{E}_\nu [1_C] &= \int_S \alpha(a) \mathbb{E}_{\mu_a} [1_C] da \\ &= \int_S \alpha(a) \mathbb{E}_\mu [1_C h_a(t, W(t))] da \\ &= \mathbb{E}_\mu \left[1_C \int_S \alpha(a) h_a(t, W(t)) da \right] \\ &= \mathbb{E}_\mu [1_C h(t, W(t))] \end{aligned}$$

□

Proposition 12. *Define*

$$\beta_{lp}(t) = W(t) - \int_0^t \frac{h'(s, W(s))}{h(s, W(s))} ds$$

where h' designates the partial derivative of h with respect to x .

Then β_{lo} is a ν_l Brownian motion and the filtration of W and β_{lp} completed with respect to ν_l are equal.

Proof: Notice that every filtration we consider here is completed with respect to ν_l . The fact that $(\beta_{lp}(t), t \in [0, 1])$ is a ν -Brownian motion is direct consequence of proposition 10 and the expression of β_{lp} gives that for $t > 1$,

$$\mathcal{F}_t^{\beta_{lp}} \subset \mathcal{F}_t^W$$

On the other hand, for $t < 1$ since $s \mapsto \frac{h'(s, x)}{h(s, x)}$ is lipschitz on $[0, t]$ and $x \mapsto \frac{h'(s, x)}{h(s, x)}$ is lipschitz on $\{x \in \mathbb{R}^n, |x| \leq k\}$ for any $k > 0$ so W is the strong solution of a stochastic differential equation relative to β_{lo} and

$$\mathcal{F}_t^W \subset \mathcal{F}_t^{\beta_{lp}}$$

W being μ_a -a.s. path continuous, we have

$$\bigcup_{t < 1} \mathcal{F}_t^W = \mathcal{F}_1^W$$

On the other hand, since $(\beta_{lo}(t), t \in [0, 1])$ is a Brownian motion, $(\langle \beta_{lo} \rangle(t), t \in [0, 1])$ is bounded by 1, so $\beta_{lo}(t)$ converges μ_{lo} -a.s. and in $L^2(\nu_l, \mathbb{R}^n)$. Denote $\beta_{lo}(1)$ the limit, for $s \in [0, 1]$ we have

$$Cov(\beta_{lo}(s), \beta_{lo}(t)) = \lim_{t' \rightarrow 1} Cov(\beta_{lo}(s), \beta_{lo}(t')) = s \wedge t'$$

and finally

$$\bigcup_{t < 1} \mathcal{F}_t^{\beta_{lo}} = \mathcal{F}_1^{\beta_{lo}}$$

$(\beta_{lo}(t), t \in [0, 1])$ is a ν_l -Brownian motion and β_{lo} and W have the same filtrations. \square

Definition 9. For $u \in G_0(\nu_l, \beta_{lo})$, we denote $\beta_{lp}^u = \beta_{lp} + u$.

Proposition 13. Set $u \in G_0(\nu_l, \beta_{lo})$, then there exists a unique ν_l -a.s. path continuous process W_{lp}^u such that

$$W_{lp}^u(t) = \beta_{lp}^u(t) + \int_0^t \frac{h'(s, W_{lo}^u(s))}{h(s, W_{lo}^u(s))} ds$$

Proof: Set $t < 1$, since $s \mapsto \frac{h'(s, x)}{h(s, x)}$ is lipschitz on $[0, t]$ for any $t < 1$ and $x \mapsto \frac{h'(s, x)}{h(s, x)}$ is lipschitz on $\{x \in \mathbb{R}^n, |x| \leq k\}$ for any $k > 0$, there exists a unique ν_l -a.s. path-continuous process $(W_{lo}^u, u \in [0, 1])$ such that for any $t < 1$,

$$W_{lp}^u(t) = \beta_{lp}^u(t) + \int_0^t \frac{h'(s, W_{lo}^u(s))}{h(s, W_{lo}^u(s))} ds$$

It remains to prove that there is no explosion in 1. Set $\tilde{\nu}_l^u$ the measure on \mathbb{W} defined by

$$\frac{d\nu_l^u}{d\nu_l} = \rho(-\delta_{\beta_{lo}} u)$$

Since $u \in G_0(\nu_l, \beta_{lo})$, it is clear that the law of W_{lp}^u under ν_l^u is the same as the law of W_{lp} under ν_l so

$$\nu_l^u \left(\limsup_{t \rightarrow 1} |W_{lp}^u(t)| = \infty \right) = \nu_l \left(\limsup_{t \rightarrow 1} |W_{lp}(t)| = \infty \right) = 0$$

$\tilde{\nu}_l^u \sim \nu_l$ so

$$\nu_l \left(\limsup_{t \rightarrow 1} |W_{lp}^u(t)| = \infty \right) = 0$$

□

Theorem 14. $(\mathbb{W}, \nu_l, \beta_{lp}, (W_{lp}^u)_{u \in \mathcal{D}})$ verify the conditions of section 2.

$(\mathbb{W}, \nu_l, \beta_{lo}, (W_{lp}^u)_{u \in G_0(\nu_l, \beta_{lo})})$ verify the conditions of definition 4.

Proof: We have $W_{lp}^0 = W$ and β_{lp} is a ν_l -Brownian motion. Now we just have to verify the conditions of definition 4 since those imply conditions (iii) to (v) of section 2.

(i), (ii) and (iii) are clear, so is (vi) taking $\tilde{\mathcal{D}} = G_0$.

Set $u \in G_0(\nu_l, \beta_{lo})$, we have

$$\begin{aligned} \beta_{lp} \circ W_{lp}^u(t) &= \left(W(t) - \int_0^t \frac{h'(s, W(s))}{h(s, W(s))} ds \right) \circ W_{lp}^u \\ &= W_{lo}^u(t) - \int_0^t \frac{h'(s, W_{lo}^u(s))}{h(s, W_{lo}^u(s))} ds \\ &= \beta_{lp}^u(t) \end{aligned}$$

so condition (iv) is verified. Now set $v \in G_0(\nu_l, \beta_{lo})$ such that $v + u \circ W^v \in G_0(\nu_l, \beta_{lo})$, we have

$$\begin{aligned} W_{lp}^u(t) \circ W_{lp}^v &= \left(\beta_{lo}^u(t) + \int_0^t \frac{h'(s, W(s))}{h(s, W(s))} ds \right) \circ W_{lp}^v \\ &= \beta_{lo}(t) + v(t) + u(t) \circ W_{lo}^v + \int_0^t \frac{h'(s, W_{lo}^u(s))}{h(s, W_{lo}^u(s))} ds \end{aligned}$$

$W^u \circ W^v$ and $W^{v+u \circ W^v}$ are both ν_l -a.s. path continuous so the uniqueness result in proposition 13 gives ν_l -a.s.

$$W^u \circ W^v = W^{v+u \circ W^v}$$

and condition (v) is verified. □

Corollary 9. It is clear that for every $u \in \mathcal{D}$, we clearly have ν_l -a.s.

$$W_{lo}^u(w) = W_{lo}^{u(w)}(w)$$

so theorem 6 applies.

Corollary 10. Theorem 8 applies. Set $u \in G_2(\nu_l, \beta_{lo})$, we have ν_l -a.s.

$$W_{lo}^u = W + \int_0^\cdot \dot{u}(t) + \frac{h'(t, W_{lo}^u(t))}{h(t, W_{lo}^u(t))} - \frac{h'(t, W(t))}{h(t, W(t))} dt$$

so

$$H(W_{lo}^u \nu_l | \nu_l) = \frac{1}{2} \mathbb{E}_{\nu_l} [|u|_H^2]$$

if and only if there exists $v \in G_0(\nu_l, \beta_{lo})$ such that W_{lo}^v is a strong solution to the stochastic differential equation:

$$dW_{lo}^v(t) = - \left(\dot{u}(t) + \frac{h'(t, W_{lo}^u(t))}{h(t, W_{lo}^u(t))} - \frac{h'(t, W(t))}{h(t, W(t))} \right) dt \circ W_{lo}^v + dW(t)$$

6.4. Diffusing particles without collision. Set $\sigma, b, \delta, \gamma \in \mathbb{R}$ such that

$$\sigma^2 \leq 2\gamma$$

The proof of the following theorem can be found in [14] or [3].

Theorem 15. Set $(\Omega, \theta, (\mathcal{G}_t))$ a filtered probability space, $(z_1(0), \dots, z_n(0)) \in \mathbb{R}^n$ and $B = (B_1, \dots, B_n)$ a \mathbb{R}^n -valued θ -Brownian motion. We consider the following stochastic differential system:

$$\begin{aligned} Z_1(t) &= z_1(0) + \sigma B_1(t) + b \int_0^t Z_1(s) ds + ct + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{1\}} \int_0^t \frac{ds}{Z_1(s) - Z_j(s)} \\ &\vdots \\ Z_n(t) &= z_n(0) + \sigma B_n(t) + b \int_0^t Z_n(s) ds + ct + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{n\}} \int_0^t \frac{ds}{Z_n(s) - Z_j(s)} \end{aligned}$$

under the condition that θ -a.s. for every $t \in [0, \infty)$

$$Z_1(t) \leq \dots \leq Z_n(t)$$

This system admits a unique strong solution on $(\Omega, \theta, (\mathcal{G}_t), B)$ and the first collision time is θ -a.s. equal to ∞ .

Consider $(\Omega, \theta, (\mathcal{G}_t))$ a filtered probability space, $(z_1(0), \dots, z_n(0)) \in \mathbb{R}^n$ and $B = (B_1, \dots, B_n)$ a \mathbb{R}^n -valued θ -Brownian motion, and Z the strong solution of the stochastic differential system of theorem 15. Denote $\nu_{pa} = Z$ the image measure of Z . For $1 \leq i \leq n$, denote W_1, \dots, W_n the coordinates of W and define

$$M_i(t) = W_i(t) - z_i(0) - b \int_0^t W_i(s) ds - ct - \gamma \sum_{j \in \{1, \dots, n\} \setminus \{i\}} \int_0^t \frac{ds}{W_i(s) - W_j(s)}$$

and

$$M = (M_1, \dots, M_n)$$

M is a local martingale and

$$\langle M_i, M_j \rangle(t) = \sigma^2 t$$

Define

$$\beta_{pa} = \frac{1}{\sigma} M$$

Levy theorem clearly ensures that β is a ν_{pa} -Brownian motion and we clearly have for every $1 \leq i \leq n$,

$$W_i(t) = z_i(0) + \sigma \beta_{pa,i}(t) + b \int_0^t W_i(s) ds + ct + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{i\}} \int_0^t \frac{ds}{W_i(s) - W_j(s)}$$

For $u \in G_0(\nu_{pa}, \beta_{pa})$ denote

$$\beta_{pa}^u = \beta_{pa} + u$$

and ν_{pa}^u the probability measure given by

$$\frac{d\nu_{pa}^u}{d\nu_{pa}} = \rho(-\delta_{\beta_{pa}} u)$$

According to Girsanov theorem, $\beta_{pa} + u$ is a Brownian motion under ν_{pa}^u , so according to theorem 15, there exists a unique ν_{pa}^u -a.s. continuous process $W_{pa}^u = (W_{pa,1}^u, \dots, W_{pa,n}^u)$ such that ν_{pa}^u -a.s. for every $1 \leq i \leq n$

$$W_{pa,i}^u(t) = z_i(0) + \sigma \beta_{pa,i}^u(t) + b \int_0^t W_{pa,i}^u(s) ds + ct + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{i\}} \int_0^t \frac{ds}{W_{pa,i}^u(s) - W_{pa,j}^u(s)}$$

and ν_{pa}^u -a.s. for every $t \in [0, 1]$

$$W_{pa,1}^u(t) \leq \dots \leq W_{pa,n}^u(t)$$

Since $\nu_{pa}^u \sim \nu_{pa}$, W^u is ν_{pa} -a.s. continuous and ν_{pa} -a.s. for every $1 \leq i \leq n$

$$W_{pa,i}^u(t) = z_i(0) + \sigma \beta_{pa,i}^u(t) + b \int_0^t W_{pa,i}^u(s) ds + ct + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{i\}} \int_0^t \frac{ds}{W_{pa,i}^u(s) - W_{pa,j}^u(s)}$$

and ν_{pa} -a.s. for every $t \in [0, 1]$

$$W_{pa,1}^u(t) \leq \dots \leq W_{pa,n}^u(t)$$

Theorem 16. $(\mathbb{W}, \nu_{pa}, \beta_{pa}, (W_{pa}^u)_{u \in \mathcal{D}})$ verify the conditions of section 2.

$(\mathbb{W}, \nu_{pa}, \beta_{pa}, (W_{pa}^u)_{u \in G_0(\nu_{pa}, \beta_{pa})})$ verify the conditions of definition 4.

Proof: (i), (ii), (iii) and (vi) are clear. Set $u \in G_0(\nu_{pa}, \beta_{pa})$, a straight calculation gives ν_{pa} -a.s.

$$\beta_{pa} \circ W_{pa}^u = \beta_{pa}^u$$

hence (iv).

Now we prove condition (v). Set $u, v \in G_0(\nu_{pa}, \beta_{pa})$ such that $v + u \circ W_{pa}^v \in G_0(\nu_{pa}, \beta_{pa})$, we have ν_{pa} -a.s.

$$\begin{aligned} & W_{pa,i}^u(t) \circ W_{pa}^v \\ = & z_i(0) \\ & + \left(\sigma (\beta_{pa,i}(t) + u(t)) + b \int_0^t W_{pa,i}^u(s) ds + ct + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{i\}} \int_0^t \frac{ds}{W_{pa,i}^u(s) - W_{pa,j}^u(s)} \right) \circ W_{pa}^v \\ = & z_i(0) + \sigma (\beta_{pa,i}^v(t) + u(t) \circ W_{pa}^v) + b \int_0^t W_{pa,i}^u(s) \circ W_{pa}^v ds + ct \\ & + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{i\}} \int_0^t \frac{ds}{W_{pa,i}^u(s) \circ W_{pa}^v - W_{pa,j}^u(s) \circ W_{pa}^v} \\ = & z_i(0) + \sigma \beta_{pa,i}^{v+u \circ W_{pa}^v}(t) + b \int_0^t W_{pa,i}^u(s) \circ W_{pa}^v ds + ct \\ & + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{i\}} \int_0^t \frac{ds}{W_{pa,i}^u(s) \circ W_{pa}^v - W_{pa,j}^u(s) \circ W_{pa}^v} \end{aligned}$$

so the uniqueness of theorem 15 gives ν_{pa} -a.s.

$$W_{pa}^u \circ W_{pa}^v = W_{pa}^{v+u \circ W_{pa}^v}$$

□

Corollary 11. *It is clear that for every $u \in \mathcal{D}$, we clearly have ν_{pa} -a.s.*

$$W_{pa}^u(w) = W_{pa}^{u(w)}(w)$$

so theorem 6 applies.

Corollary 12. *Theorem 8 applies. Set $u \in G_2(\nu_{pa}, \beta_{pa})$, for $i \in \{1, \dots, n\}$, define*

$$\dot{w}_{pa,i}^u(t) = \dot{u}_i(t) + b(W_{pa,i}^u(t) - W(t)) + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{i\}} \left(\frac{1}{W_{pa,i}^u(t) - W_{pa,j}^u(t)} - \frac{1}{W_i(t) - W_j(t)} \right)$$

We have ν_{pa} -a.s.

$$W_{pa}^u = I_{\mathbb{W}} + w_{pa}^u$$

so

$$H(W_{pa}^u \nu_{pa} | \nu_{pa}) = \frac{1}{2} \mathbb{E}_{\nu_{pa}} [|u|_H^2]$$

if and only if there exists $v \in G_0(\nu_{pa}, \beta_{pa})$ such that W_{pa}^v is a strong solution to the stochastic differential system:

$$\begin{aligned} dW_{pa,1}^v(t) &= \dot{w}_{pa,1}^u(t) dt \circ W_{pa}^v + dW_1(t) \\ &\vdots \\ dW_{pa,n}^v(t) &= \dot{w}_{pa,n}^u(t) dt \circ W_{pa}^v + dW_n(t) \end{aligned}$$

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